Topic 11

University Mechanical Engineering

What this course is, and how to use it

This is the eleventh course in University Mechanical Engineering: **theory of machines** and **vibration** – how mechanisms move, and how machines shake. Two machines from the cooling loop carry the course: a small reciprocating dosing pump (a slider-crank, 300 rpm) that adds inhibitor to the water, and the main pump (120 kg, 1440 rpm) on its rubber mounts and steel base frame.

Scope limit, stated once: a study course, not a degree; there is no test rig. The "measured" decay record is a worked set of numbers, mount and frame stiffnesses are typical values, and every number printed here is recomputed in the course suite and checked by a second, independent method (a numerical derivative, an energy check or a direct solve).

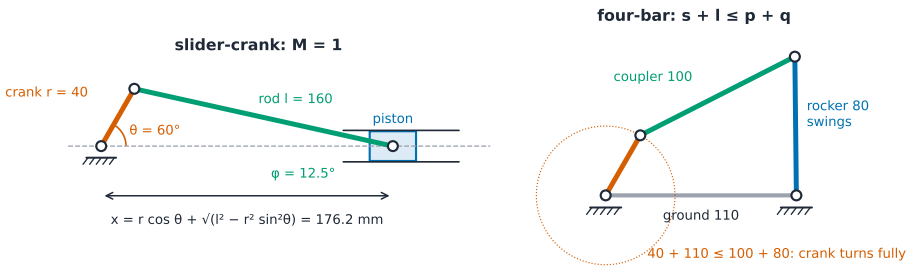
- Units 11.1–11.2 cover mechanism position, velocity and acceleration.
- Unit 11.3 balances the rotating and reciprocating masses.
- Units 11.4–11.6 cover free, forced and multi-degree-of-freedom vibration.

The habit: **draw the mechanism or the free-body diagram to scale first** – a sketch catches a wrong sign before the algebra does.

Mechanisms and kinematic pairs

Why it matters. The dosing pump turns rotation into a stroke of exactly the right length. Choosing the links, and knowing how many inputs the mechanism needs, comes before any force calculation.

The idea, with a picture. A **mechanism** is a chain of **links** joined by **kinematic pairs**: lower pairs (pin, slider) allow one relative motion; higher pairs (cam, gear contact) allow two. Its **degrees of freedom** come from **Gruebler's equation** for planar mechanisms: $M = 3(n-1) - 2j_1 - j_2$, with n links (ground included), j_1 one-freedom joints and j_2 two-freedom joints.



The dosing pump's slider-crank at 60°, and a Grashof crank-rocker four-bar

The derivation (loop closure). Write the links as vectors that close a loop. For the slider-crank (crank r, rod l, piston at x): $re^{i\theta} + le^{-i\phi} = x$. The imaginary part gives $r\sin\theta = l\sin\phi$; the real part gives $x = r\cos\theta + \sqrt{l^2 - r^2\sin^2\theta}$. For a four-bar the same loop gives two equations in the two unknown angles. The **Grashof condition** – shortest + longest ≤ sum of the other two – decides whether any link can turn fully; changing which link is ground gives the **inversions** of the chain (crank-rocker, double-crank, double-rocker).

> **Worked example 1 – the dosing pump's position.** $r = 40$ mm, $l = 160$ mm, $\theta = 60°$: $\sin\phi = 40\sin 60°/160$, so $\phi = \mathbf{12.5}°$ and $x = 20 + 156.2 = \mathbf{176.2}$ mm. The stroke is $2r = \mathbf{80}$ mm. Check by counting: 4 links, 4 pin or slider joints: $M = 3\times3 - 2\times4 = \mathbf{1}$ – one motor drives it.

> **Worked example 2 – a four-bar for a damper vane.** Links 40 (crank), 100 (coupler), 80 (rocker), 110 (ground): $s + l = 40 + 110 = 150 \le p + q = 180$, Grashof, and with the shortest link as the crank it is a **crank-rocker**. Solving the loop at $\theta_2 = 60°$ gives the coupler at $\theta_3 = \mathbf{27.0}°$ and the rocker at $\theta_4 = \mathbf{90.6}°$. Ground the coupler instead and the same chain becomes a double-rocker. A five-bar with 5 pins has $M = 3\times4 - 2\times5 = 2$ – it needs two inputs.

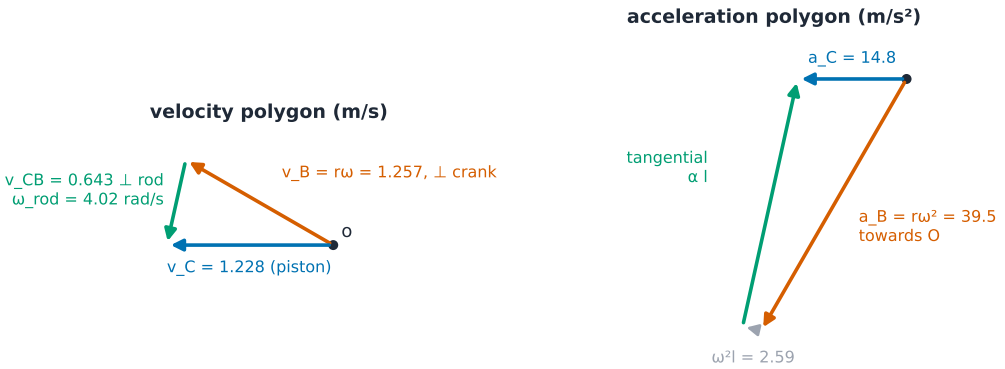
Common mistakes. Forgetting to count the ground as a link. Counting a joint where three links meet as one joint (it is two). Taking the wrong branch of the square root (the "crossed" assembly).

Summary. $M = 3(n-1) - 2j_1 - j_2$; loop closure gives position; Grashof $s + l \le p + q$ means a link can turn fully; inversions change the ground link.

Velocity and acceleration analysis

Why it matters. The piston's acceleration sets the inertia force, and that force loads the rod, the bearings and the frame. Velocity comes first, because acceleration terms use it.

The idea, with a picture. For two points on one rigid link, $\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{CB}$: the relative velocity is perpendicular to the link. Drawing these vectors from one pole gives the **velocity polygon**. Acceleration adds a centripetal part $\omega^2 r$ along the link and a tangential part αr across it: the **acceleration polygon**. An **instant centre** is a point about which a link is momentarily turning.



Velocity and acceleration polygons of the slider-crank at 60° and 300 rpm

The derivation (the Coriolis term). A point P sliding along a link that is itself rotating at ω has, in the fixed frame, $\mathbf{a}_P = \mathbf{a}_{\text{link point}} + \mathbf{a}_{rel} + 2\boldsymbol{\omega} \times \mathbf{v}_{rel}$. The last term is the **Coriolis component** : the sliding velocity is turned by the rotation, and the radius changes while the link turns – each effect gives ωv_{rel}. It appears whenever a slider runs on a rotating link (a quick-return mechanism), never on a fixed guide.

> **Worked example 1 – velocities at 60°.** $\omega = 300 \times 2\pi/60 = \mathbf{31.42}$ rad/s. Crank pin: $v_B = r\omega = \mathbf{1.257}$ m/s, perpendicular to the crank. The polygon closes with v_C horizontal and v_{CB} perpendicular to the rod: $v_C = \mathbf{1.228}$ m/s, $v_{CB} = 0.643$ m/s, so the rod turns at $\omega_3 = v_{CB}/l = \mathbf{4.02}$ rad/s. Check: the instant centre of the rod lies where the crank line meets the vertical through the piston; distances 0.312 m to B and 0.305 m to C give $1.257/0.312 = 1.228/0.305 = 4.02$ rad/s. A numerical derivative of $x(\theta)$ agrees to four figures.

> **Worked example 2 – accelerations and Coriolis.** Crank pin: $a_B = r\omega^2 = \mathbf{39.5}$ m/s^2 towards O. Rod: centripetal $\omega_3^2 l = 2.59$ m/s^2 from C towards B, tangential unknown; piston horizontal. Closing the polygon: $a_C = \mathbf{14.8}$ m/s^2. The approximate formula $a_C \approx -r\omega^2(\cos\theta + \cos 2\theta/n)$ with $n = l/r = 4$ gives 14.80 – the same. In a quick-return mechanism where a block slides at 0.6 m/s along a slotted arm turning at 5 rad/s, the Coriolis component is $2 \times 5 \times 0.6 = \mathbf{6.0}$ m/s^2, perpendicular to the arm.

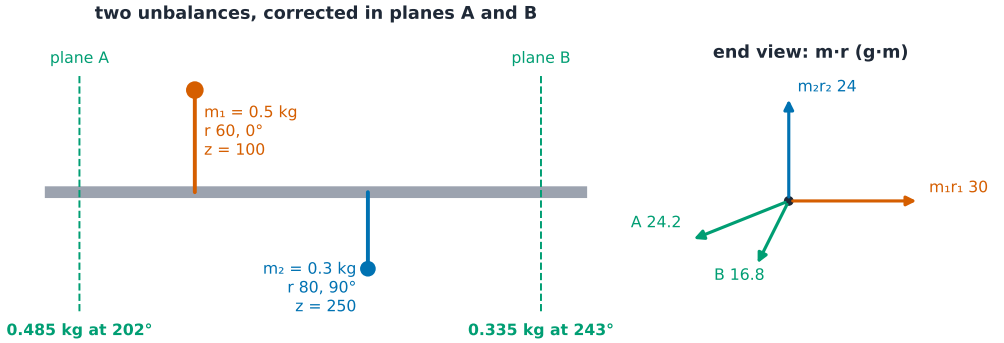
Common mistakes. Drawing relative velocity along the link instead of across it. Dropping the centripetal term of a link that rotates. Forgetting Coriolis, or adding it on a fixed slide.

Summary. $\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{CB}$; a adds $\omega^2 r$ along and αr across; Coriolis $2\omega v_{rel}$ for sliding on a rotating link; instant centres give a quick check.

Balancing

Why it matters. A residual unbalance on the main pump shaft at 1440 rpm shakes the frame, loads the bearings and fatigues the pipework. Balancing removes the cause instead of fighting the effect.

The idea, with a picture. A mass m at radius r on a shaft turning at ω pulls outwards with $mr\omega^2$. **Static balance** makes the vector sum $\sum m_i r_i = 0$; **dynamic balance** also makes the moment sum $\sum m_i r_i z_i = 0$, which takes correction masses in **two planes** . A **reciprocating mass** gives a **primary force** $mr\omega^2 \cos\theta$ and a smaller **secondary force** $mr\omega^2 \cos 2\theta/n$, which a rotating counterweight cannot cancel completely.



Two-plane balancing of the pump shaft: side view and end-view $m \cdot r$ vectors

The derivation (two planes are enough). Any set of rotating unbalances is equivalent to one force and one couple. Take moments about plane A to find the correction in plane B that cancels the couple: $m_B r_B z_B = -\sum m_i r_i z_i$ (as vectors). Then the force balance gives plane A: $m_A r_A = -\sum m_i r_i - m_B r_B$. Two unknown vectors, two vector equations.

> **Worked example 1 – the pump shaft.** $m_1 = 0.5$ kg at 60 mm, 0°, $z = 100$ mm; $m_2 = 0.3$ kg at 80 mm, 90°, $z = 250$ mm. Correction planes at $z = 0$ (A) and 400 mm (B), radius 50 mm. Moments about A: $m_B r_B = (7.5, 15.0)/0.4$ reversed $\rightarrow$ 16.8 g · m, so **0.335** kg at **243**°. Forces: $m_A r_A = 24.2$ g · m, **0.485** kg at **202**°. Before balancing, the shaft shook the bearings with $38.4\ \text{g}\cdot\text{m} \times 150.8^2 = \mathbf{873}$ N at 1440 rpm.

> **Worked example 2 – the dosing pump's reciprocating mass.** Piston and part of the rod: 0.8 kg at $r = 40$ mm, 300 rpm, $n = 4$. Primary force amplitude $mr\omega^2 = \mathbf{31.6}$ N; secondary **7.9** N at twice the speed. A counterweight balancing half the reciprocating mass leaves 15.8 N along the stroke but adds 15.8 N across it: the trade-off is a rotating force of constant size instead of a large one in one direction. The secondary force needs a second shaft at twice speed, or is simply accepted at this size.

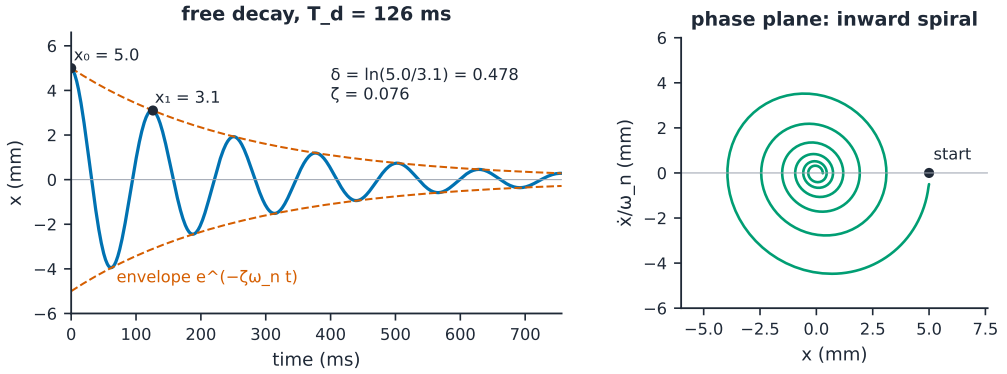
Common mistakes. Balancing statically and calling it done (the couple remains). Adding mr values as numbers, not vectors. Expecting a counterweight to cancel a reciprocating force completely.

Summary. $\sum mr = 0$ static, $\sum mrz = 0$ dynamic; two planes always suffice for rotors; reciprocating primary $mr\omega^2\cos\theta$, secondary $/n$ at 2ω; partial balance trades direction for size.

Free vibration

Why it matters. Every mounted machine is a mass on a spring. Its **natural frequency** decides whether running speed will excite it, and its damping decides how quickly a knock dies away.

The idea, with a picture. For a mass m, stiffness k and viscous damping c: $m\ddot{x} + c\dot{x} + kx = 0$. Write $\omega_n = \sqrt{k/m}$ and the **damping ratio** $\zeta = c/(2\sqrt{km})$. Below $\zeta = 1$ (**underdamped**) the mass oscillates at $\omega_d = \omega_n\sqrt{1-\zeta^2}$ inside a decaying envelope; at $\zeta = 1$ (critical) it returns fastest without overshoot; above 1 it creeps back.



Measured free decay of the pump on its mounts, and the same motion in the phase plane

The derivation (damping from a record). The underdamped solution is $x = Xe^{-\zeta\omega_n t}\cos(\omega_d t - \psi)$. Two peaks one period apart have the ratio $e^{\zeta\omega_n T_d}$, so the **logarithmic decrement** is $\delta = \ln(x_0/x_1) = 2\pi\zeta/\sqrt{1-\zeta^2}$, and $\zeta = \delta/\sqrt{4\pi^2+\delta^2}$. In the **phase plane** (x against $\dot{x}/\omega_n$) the motion is an inward spiral; without damping it would be a circle.

> **Worked example 1 – the pump on its mounts.** $m = 120$ kg on four mounts of 75 kN/m: $k = 300$ kN/m, $\omega_n = \sqrt{300\,000/120} = \mathbf{50.0}$ rad/s, $f_n = \mathbf{7.96}$ Hz; static deflection $mg/k = 3.9$ mm. A tap test gives successive peaks of 5.0 and 3.1 mm: $\delta = \ln(5.0/3.1) = \mathbf{0.478}$, $\zeta = \mathbf{0.076}$, $c = 2\zeta\sqrt{km} = 910$ N · s/m; $\omega_d = 49.86$ rad/s – damping barely changes the frequency.

> **Worked example 2 – a torsional system.** The output shaft (C45, $G = 80$ GPa, Ø30, 0.6 m long) joins the 136-tooth gear ($J_1 = 0.64$ kg · m^2, a steel disc Ø408 × 30) to the pump impeller ($J_2 = 0.12$ kg · m^2). Shaft stiffness $k_t = GJ_p/L = \mathbf{10.6}$ kN · m/rad. Two discs on a shaft vibrate against each other at $\omega = \sqrt{k_t(J_1+J_2)/(J_1J_2)} = \mathbf{324}$ rad/s (**51.6** Hz, 3090 rpm), with a node 95 mm from the gear. The shaft runs at 180 rpm: far below, so no torsional resonance.

Common mistakes. Measuring δ from peaks that are not one period apart. Using f_n in rad/s. Treating a torsional two-disc system as one disc on a fixed shaft.

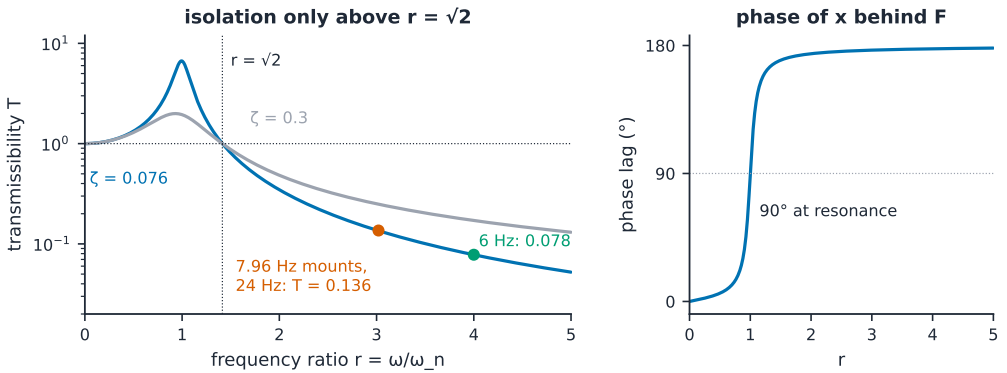
Summary. $\omega_n = \sqrt{k/m}$, $\zeta = c/2\sqrt{km}$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$; $\delta = \ln(x_0/x_1)$; two-disc torsion $\omega = \sqrt{k_t(J_1+J_2)/J_1J_2}$.

Forced vibration

Why it matters. The pump's residual unbalance pushes on its mounts 24 times a second. Whether the floor feels more or less than that push depends only on the frequency ratio and the damping.

The idea, with a picture. Under a harmonic force $F_0\sin\omega t$ the steady response has amplitude $X = \frac{F_0/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ with $r = \omega/\omega_n$, lagging the force by $\phi = \arctan\frac{2\zeta r}{1-r^2}$. At **resonance** ($r = 1$) the response is limited only by damping, and the phase lag is 90°. The **transmissibility**, the force reaching the floor over the force applied, is

$$T = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$



Transmissibility and phase against frequency ratio, with the pump's operating point

The derivation (why isolation starts at √2). Setting $T = 1$ gives $(1 - r^2)^2 = 1$, so $r = \sqrt{2}$ for every ζ. Below it the mounts amplify; above it they isolate. Physically, below resonance the spring force follows the push; well above it the mass hardly moves, so the spring has little to pass on. Damping lowers the peak but raises T above $\sqrt{2}$ – **vibration isolation** wants low damping and a low natural frequency.

> **Worked example 1 – the present mounts.** Residual unbalance after balancing: $me = 1.2\ \mathrm{g \cdot m}$; at 1440 rpm ($\omega = 150.8$ rad/s) $F_0 = me\,\omega^2 = \mathbf{27.3}$ N. With $f_n = 7.96$ Hz, $r = 24/7.96 = \mathbf{3.02}$ and $\zeta = 0.076$: $T = \mathbf{0.136}$, so 3.7 N reaches the floor and the pump moves **11** µm. Before balancing (873 N) the floor would have felt 118 N. During start-up the pump passes through resonance: there $X = me/(2\zeta m) = 66$ µm – run up quickly.

> **Worked example 2 – designing for 90% isolation.** $T \le 0.1$ at $\zeta = 0.076$ needs $r \ge 3.40$, so $f_n \le 7.06$ Hz. Choose $f_n = 6$ Hz: $k = m(2\pi \times 6)^2 = \mathbf{170.5}$ kN/m (four mounts of 42.6 kN/m), static deflection **6.9** mm. Then $r = 4.0$, $T = \mathbf{0.078}$, 2.1 N to the floor. The cost: softer mounts let the pump sit lower and move more when a pipe pulls on it, so flexible pipe connections become part of the design.

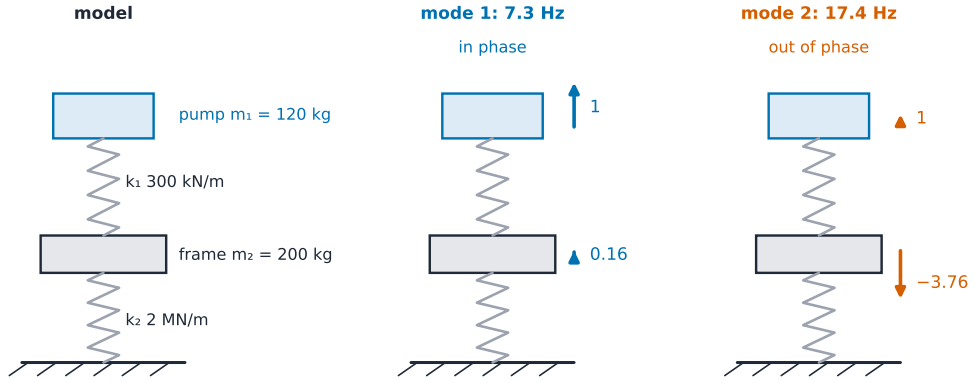
Common mistakes. Putting f_n near the running speed. Adding damping to "improve" isolation above $\sqrt{2}$. Forgetting the passage through resonance at start-up.

Summary. $X = (F_0/k)/\sqrt{(1 - r^2)^2 + (2\zeta r)^2}$; 90° lag at resonance; isolation needs $r > \sqrt{2}$, ideally $r \ge 3$; softer mounts trade static deflection for isolation.

Multi-degree-of-freedom systems

Why it matters. The pump sits on a steel base frame that itself rests on the floor. Two masses and two springs give two natural frequencies – and either can be in the wrong place.

The idea, with a picture. Newton's law for each mass gives $\mathbf{M\ddot{x}} + \mathbf{C\dot{x}} + \mathbf{Kx} = \mathbf{F}$, with mass, damping and stiffness matrices. Trying $\mathbf{x} = \mathbf{u}\sin\omega t$ without damping gives the **eigenvalue problem** $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{u} = 0$: each root ω_i is a natural frequency and each vector $\mathbf{u}_i$ a **mode shape** .



The pump on its base frame as two degrees of freedom, and its two mode shapes

The derivation (why modes simplify everything). Mode shapes are orthogonal through the mass and stiffness matrices: $\mathbf{u}_i^T\mathbf{M}\mathbf{u}_j = 0$ for $i \ne j$. Writing $\mathbf{x} = \sum q_i(t)\mathbf{u}_i$ therefore splits the coupled equations into separate single-degree equations, one per mode – **mode superposition** . Each mode responds like the SDOF system of unit 11.5, and the total response is their sum.

> **Worked example 1 – the pump on its frame.** $m_1 = 120$ kg (pump), $k_1 = 300$ kN/m (mounts), $m_2 = 200$ kg (frame), $k_2 = 2$ MN/m (frame feet). $\mathbf{M} = \mathrm{diag}(120, 200)$, $\mathbf{K} = \begin{bmatrix} 300 & -300 \\ -300 & 2300 \end{bmatrix}$ kN/m. $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ gives $\omega_1 = \mathbf{45.8}$ rad/s (**7.30** Hz) with $\mathbf{u}_1 = (1, 0.160)$ – pump and frame in phase, the mounts doing the moving – and $\omega_2 = \mathbf{109.1}$ rad/s (**17.4** Hz) with $\mathbf{u}_2 = (1, -3.76)$ – the frame bouncing against the pump. The frame lowers the first frequency from 7.96 to 7.30 Hz.

> **Worked example 2 – the response at 24 Hz.** Solving $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{X} = (27.3, 0)$ N directly at $\omega = 150.8$ rad/s: pump **11.4** µm, frame **1.5** µm, floor force $k_2 X_2 = \mathbf{3.0}$ N – close to the SDOF estimate (3.4 N undamped), because 24 Hz is above both modes. Mode superposition gives the same numbers, mode by mode. If the frame were lighter and stiffer feet put mode 2 near 24 Hz, the floor force would rise sharply: check every mode, not only the first.

Common mistakes. Writing $\mathbf{K}$ with the wrong sign on the coupling terms. Scaling a mode shape and thinking it changed. Checking only the lowest mode against the running speed.

Summary. $\mathbf{M\ddot{x}} + \mathbf{Kx} = \mathbf{F}$; $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$; mode shapes are orthogonal, so superposition turns n coupled equations into n single ones; keep every mode clear of

the forcing frequency.