Topic 10

University Mechanical Engineering

What this course is, and how to use it

This is the tenth course in University Mechanical Engineering: **heat transfer** – how fast heat moves, not only how much (Course 5 counted the energy; this course finds the rate). We follow the cooling loop again: its warm pipes, the parts it quenches, the exchanger that cools its water and the electronics that drive its pump.

Scope limit, stated once: a study course, not a degree; there is no laboratory. Property values are typical textbook values at the stated temperatures, correlations are quoted with their ranges, and every number printed here is recomputed in the course suite with an energy balance closed on each system.

- Units 10.1–10.2 cover conduction, steady and transient.
- Units 10.3–10.4 cover convection and radiation.
- Units 10.5–10.6 apply all three to exchangers and to a cooling design.

The habit: **draw the resistance network before writing any equation** – heat flows like current, $q = \Delta T / \sum R$, and the largest resistance controls the answer.

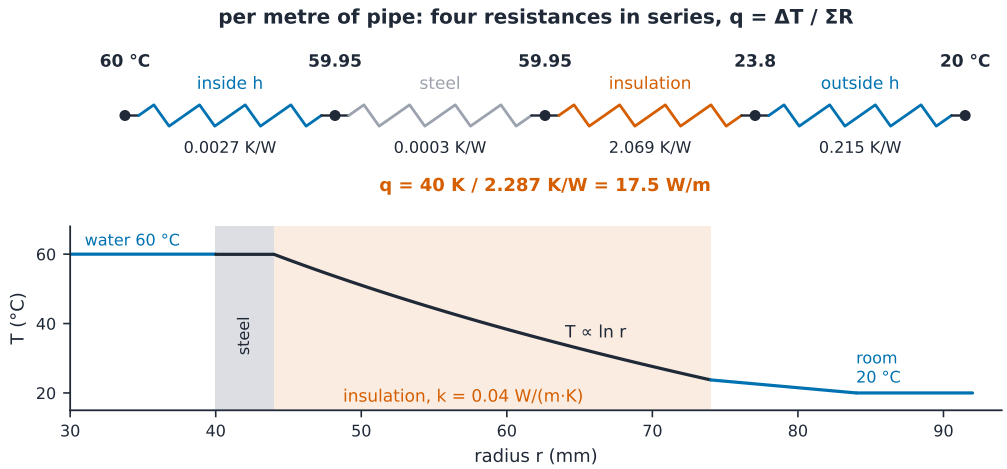
Conduction

Why it matters. The loop's hot-water branch runs at 60 °C through a room at 20 °C. Insulating it is cheap; deciding how much insulation, and whether it helps at all, takes a conduction model.

The idea, with a picture. Fourier's law says heat flux is proportional to the temperature gradient: $q'' = -k\,dT/dx$, with **thermal conductivity** k. In steady flow through layers in series, each layer is a **thermal resistance**:

- plane wall: $R = L/(kA)$;
- cylindrical wall, per length ℓ: $R = \ln(r_2/r_1)/(2\pi k\ell)$;
- convection at a surface: $R = 1/(hA)$.

Resistances in series add, like electrical ones, and the temperature drop across each is qR.



The insulated pipe as four resistances in series, and the temperature falling through the wall

The derivation (why a cylinder gives a logarithm). In steady state the same heat q crosses every cylinder of radius r: $q = -k(2\pi r\ell)\,dT/dr$. Separate and integrate from r_1 to r_2: $T_1 - T_2 = \frac{q}{2\pi k\ell}\ln\frac{r_2}{r_1}$. The area grows with r, so the gradient falls with r and the profile curves as $\ln r$. If k varies linearly with temperature, $k = k_0(1+\beta T)$, the same steps work with the **mean conductivity** at the average temperature: for mineral wool $k_0 = 0.035$, $\beta = 0.004$/K between 20 and 60 °C, $k_m = 0.035(1 + 0.004 \times 40) = 0.0406$ W/(m · K).

> **Worked example 1 – a composite wall.** A pump-room wall, 12 m²: brick 200 mm ($k = 0.70$), mineral wool 50 mm ($k = 0.04$), plaster 12 mm ($k = 0.5$); $h_i = 8$, $h_o = 25$ W/(m² · K); inside 22 °C, outside −5 °C. Resistances (K/W): 0.0104, 0.0238, **0.1042**, 0.0020, 0.0033; total **0.1437** K/W. So $q = 27/0.1437 =$ **188** W. The wool, a quarter of the thickness, takes 73% of the temperature drop: 20.0 → 15.6 → **−4.0** °C across it.

> **Worked example 2 – the insulated pipe.** Water 60 °C, steel pipe $r_1 = 40$, $r_2 = 44$ mm ($k = 50$), insulation to $r_3 = 74$ mm ($k = 0.04$), $h_i = 1500$, $h_o = 10$ W/(m² · K), room 20 °C. Per metre: $R = 0.0027 + 0.0003 +$ **2.069** $+ 0.215 = 2.287$ K/W, so $q = 40/2.287 =$ **17.5** W/m and the outer surface sits at 23.8 °C. The bare pipe loses **110** W/m by convection alone: the insulation saves 84%. (The **critical radius** $r_c = k/h = 4$ mm is far below 44 mm, so adding insulation here can only reduce the loss.)

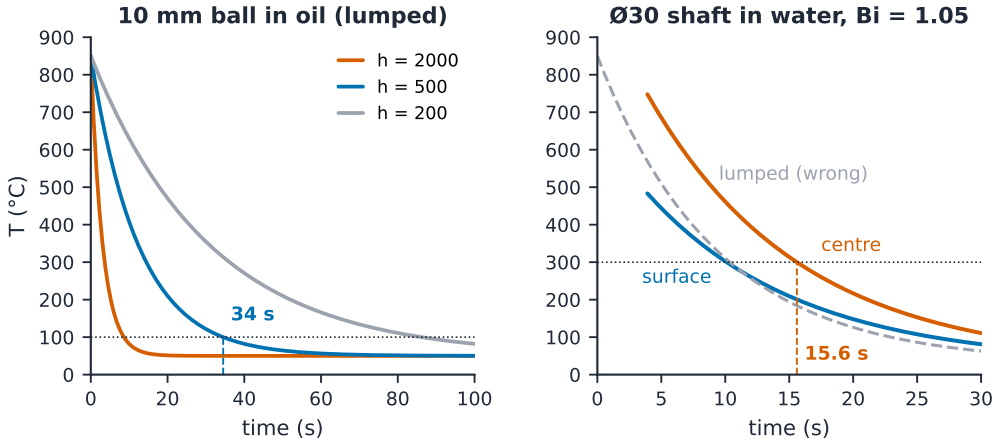
Common mistakes. Using $L/(kA)$ for a thick cylinder. Forgetting the surface (convection) resistances. Adding conductances instead of resistances for layers in series.

Summary. $q = \Delta T/\sum R$; $R = L/kA$, $\ln(r_2/r_1)/2\pi k\ell$, $1/hA$; variable k → use k at the mean temperature; insulation helps only above $r_c = k/h$.

Transient conduction

Why it matters. Hardening a bearing ball or a shaft means quenching it from 850 °C. How long it must stay in the bath, and whether the centre cools in time, are transient questions.

The idea, with a picture. The **Biot number** $Bi = hL_c/k$ (with $L_c = V/A_s$) compares the internal conduction resistance to the surface convection resistance. If $Bi < 0.1$, the body is nearly uniform inside and the **lumped capacitance** model holds: $\dfrac{T - T_\infty}{T_i - T_\infty} = e^{-t/\tau}$, $\tau = \rho c V/(hA_s)$. If not, the centre lags the surface, and we need the one-dimensional solution in terms of Bi and the **Fourier number** $Fo = \alpha t/L^2$.



Lumped cooling of a small ball, and the centre lagging the surface of a quenched shaft

The derivation (the lumped model). If the body has one temperature $T(t)$, the heat lost at the surface empties its stored energy: $-\rho c V\, dT/dt = hA_s(T - T_\infty)$. Separate and integrate: $\ln\frac{T-T_\infty}{T_i-T_\infty} = -\frac{hA_s}{\rho cV}t$. For a long body with $Bi > 0.1$, the exact series solution keeps only its first term once $Fo > 0.2$: for a cylinder of radius r_o, $\theta_0 = C_1 e^{-\zeta_1^2 Fo}$ at the centre and $\theta = \theta_0 J_0(\zeta_1 r/r_o)$ elsewhere, with ζ_1 and C_1 from tables (the "charts" are plots of exactly this).

> **Worked example 1 – a quenched bearing ball.** Steel ball, 10 mm ($\rho = 7850$, $c = 475$, $k = 43$), oil at 50 °C, $h = 500$: $L_c = r/3 = 1.67$ mm, $Bi = \mathbf{0.019}$ – lumped is valid. $\tau = 7850 \times 475 \times 0.00167/500 = \mathbf{12.4}$ s; to cool from 850 to 100 °C: $t = 12.4\ln(800/50) = \mathbf{34}$ s. A still bath ($h = 200$) takes 86 s; a violent one ($h = 2000$) 8.6 s.

> **Worked example 2 – the Ø30 shaft in water.** $h = 3000$, water 30 °C, from 850 °C: $Bi = hr_o/k = \mathbf{1.05}$ – too large for lumped. Tables give $\zeta_1 = 1.278$, $C_1 = 1.215$. For the centre to reach 300 °C: $\theta_0 = 270/820 = 0.329$, $Fo = \ln(C_1/\theta_0)/\zeta_1^2 = \mathbf{0.80}$, $t = Fo\, r_o^2/\alpha = \mathbf{15.6}$ s, and the surface is then at about **200** °C. The lumped model would say 10.4 s – 33% too short, and it cannot see the 100 K difference across the shaft.

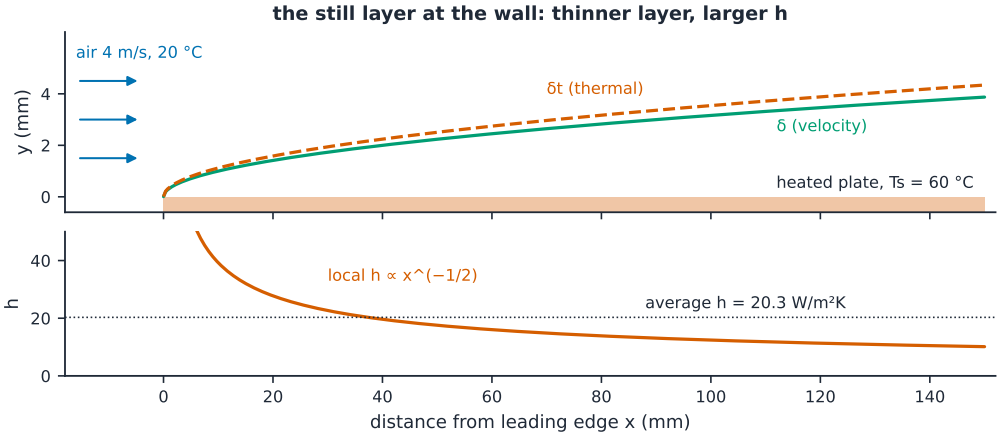
Common mistakes. Skipping the Bi check. Using $L_c = r$ in the lumped model (it is V/A_s: $r/3$ for a sphere, $r/2$ for a long cylinder). Using the one-term solution when $Fo < 0.2$.

Summary. $Bi = hL_c/k$; below 0.1, $\theta = e^{-t/\tau}$ with $\tau = \rho cV/hA_s$; otherwise one-term solutions in Bi and Fo.

Convection

Why it matters. Almost every number h used so far was given. In practice h must come from the flow, and it can change by a factor of a thousand between still air and fast water.

The idea, with a picture. At the wall the fluid is still, so heat crosses the first thin layer by conduction; the **thermal boundary layer** sets how thin that layer is. Thinner layer, larger h. Experiments are packed into the **Nusselt number** $Nu = hL/k_f$, written as a correlation in the **Reynolds number** and the **Prandtl number** $Pr = \nu/\alpha$ – each valid only in its stated range.



Velocity and thermal boundary layers on a heated plate; local h falls as the layer thickens

The derivation (why Nu is h made dimensionless). At the wall, Fourier's law in the fluid equals Newton's law of cooling: $-k_f\, \partial T/\partial y|_0 = h(T_s - T_\infty)$. Divide by $k_f(T_s - T_\infty)/L$: $Nu = hL/k_f$ is the dimensionless wall gradient. Course 7's boundary

layer grew as $\delta \propto x/\sqrt{Re_x}$; the thermal layer follows it, $\delta_t \approx \delta/Pr^{1/3}$, so $h \propto 1/\delta_t \propto x^{-1/2}$ in laminar flow.

> **Worked example 1 – air over a plate (laminar).** Air 4 m/s, 20 °C ($\nu = 1.6 \times 10^{-5}$, $k = 0.0265$, $Pr = 0.71$) along a 150 mm plate: $Re_L = \mathbf{37\,500}$ (laminar, below $5{\times}10^5$). Average: $Nu = 0.664 Re_L^{1/2} Pr^{1/3} = \mathbf{114.7}$, so $h = 114.7{\times}0.0265/0.15 = \mathbf{20.3}$ W/(m^2 · K). The layers are thin: $\delta = 3.9$ mm, $\delta_t = 4.3$ mm at the trailing edge.

> **Worked example 2 – the loop water inside the pipe.** 16.8 L/s in the 80 mm pipe at 40 °C ($\nu = 0.658 \times 10^{-6}$, $k = 0.631$, $Pr = 4.32$): $V = 3.34$ m/s, $Re = \mathbf{4.06 \times 10^5}$, turbulent. **Dittus–Boelter** ($Re > 10^4$, $0.6 < Pr < 160$, heating): $Nu = 0.023 Re^{0.8} Pr^{0.4} = \mathbf{1268}$, $h \approx \mathbf{10\,000}$ W/(m^2 · K) – five hundred times the air value, which is why the inside resistance in unit 10.1 was negligible.

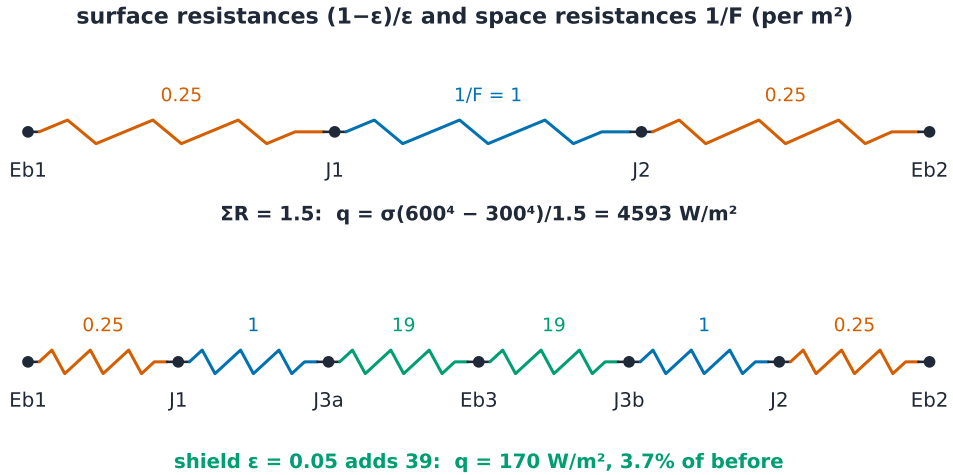
Common mistakes. Using a correlation outside its range (laminar formula at $Re = 10^6$). Evaluating properties at the wrong temperature (use the film or bulk mean). Using pipe diameter in an external-flow Reynolds number for a plate.

Summary. $Nu = hL/k_f$; laminar plate $Nu = 0.664 Re^{1/2} Pr^{1/3}$; turbulent pipe $Nu = 0.023 Re^{0.8} Pr^{0.4}$; check the range first.

Radiation

Why it matters. In unit 10.1 the bare pipe lost 110 W/m by convection. That figure is wrong: a 60 °C surface also radiates, and ignoring it underestimates the loss by a third.

The idea, with a picture. Every surface emits $\varepsilon\sigma T^4$ ($\sigma = 5.67 \times 10^{-8}$ W/(m^2 · K^4), **emissivity** ε, absolute temperature). Between surfaces, the **view factor** F_{ij} is the fraction of radiation leaving i that reaches j. For **grey surfaces** each surface adds a surface resistance $(1-\varepsilon)/(\varepsilon A)$ and each pair a space resistance $1/(A_i F_{ij})$; the **radiosity** J is the node between them.



Radiation network between two grey plates, without and with a low-emissivity shield

The derivation (exchange between two surfaces). Driving "potentials" are $E_b = \sigma T^4$; for two surfaces that see only each other, the three resistances are in series:

$$q_{12} = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

For large parallel plates ($F = 1$, equal areas) per m^2 this becomes $\sigma(T_1^4 - T_2^4)/(1/\varepsilon_1 + 1/\varepsilon_2 - 1)$. For a small body in a large room it becomes $\varepsilon_1 \sigma A_1 (T_1^4 - T_2^4)$. View factors obey reciprocity $A_i F_{ij} = A_j F_{ji}$ and summation $\sum_j F_{ij} = 1$; for two coaxial discs of radius 100 mm, 200 mm apart, $F_{12} = 0.172$.

> **Worked example 1 – the bare pipe, honestly.** Linearise radiation as $h_r = \varepsilon\sigma(T_s^2 + T_{sur}^2)(T_s + T_{sur})$. With $\varepsilon = 0.8$ and $T_s = 59.5$ °C: $h_r = \mathbf{5.58}$ W/(m^2 · K), beside $h_c = 10$. Re-solving the network with $h = h_c + h_r$ in parallel: $q = \mathbf{170}$ W/m (convection 109, radiation **61**). Insulated, the surface is at 22.7 °C and $q = \mathbf{18.0}$ W/m. The saving is 89%, larger than the 84% found without radiation.

> **Worked example 2 – a radiation shield.** An oven wall at 600 K faces a casing at 300 K; both $\varepsilon = 0.8$. Per m^2: $q = \sigma(600^4 - 300^4)/(1/0.8 + 1/0.8 - 1) = \mathbf{4593}$ W/m^2. A thin polished shield ($\varepsilon = 0.05$ both sides) between them adds $2/0.05 - 1 = 39$ to the sum (two surface resistances of 19 and a space resistance of 1): $q = \mathbf{170}$ W/m^2, 3.7% of before, and the shield settles at 512 K.

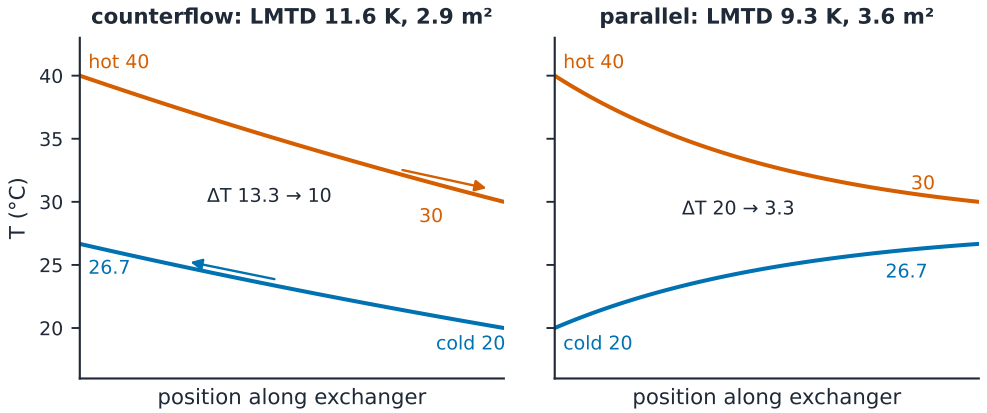
Common mistakes. Using °C in T^4. Ignoring radiation at modest temperatures (here it was 36% of the bare-pipe loss). Adding emissivity fractions instead of resistances.

Summary. $E_b = \sigma T^4$; surface $(1-\varepsilon)/\varepsilon A$, space $1/A_i F_{ij}$; h_r adds in parallel with h_c; a shield adds two surface resistances.

Heat exchangers

Why it matters. The loop rejects its heat to plant water through a **plate heat exchanger** . Size it too small and the pump motor overheats; too large and money is wasted.

The idea, with a picture. The duty is $Q = UA\,\Delta T_{lm}$, with the **overall heat transfer coefficient** U and the **log mean temperature difference** $\Delta T_{lm} = (\Delta T_1 - \Delta T_2)/\ln(\Delta T_1/\Delta T_2)$, where ΔT_1, ΔT_2 are the end differences. **Counterflow** keeps the two streams closer in temperature along the whole length than **parallel flow** , so it needs less area for the same duty. When outlets are unknown, the **effectiveness–NTU method** – is easier.



Temperatures along a counterflow and a parallel-flow exchanger doing the same duty

The derivation (effectiveness). The largest possible duty is $Q_{max} = C_{min}(T_{h,i} - T_{c,i})$, with heat capacity rates $C = \dot{m}c_p$. The **effectiveness** is $\varepsilon = Q/Q_{max}$, and integrating the two energy balances along the length gives, for counterflow, with $NTU = UA/C_{min}$ and $C_r = C_{min}/C_{max}$: $\varepsilon = \dfrac{1 - e^{-NTU(1-C_r)}}{1 - C_r e^{-NTU(1-C_r)}}$.

Worked example 1 – sizing by LMTD. Loop water 2.0 kg/s cooled from 40 to 30 °C: $Q = 2.0 \times 4180 \times 10 =$ **83.6** kW. Plant water 3.0 kg/s enters at 20 °C and leaves at 26.7 °C. Counterflow ends: 13.3 and 10 K, $\Delta T_{lm} =$ **11.6** K. With $U = 2500$ W/(m² · K): $A = 83\,600/(2500 \times 11.6) =$ **2.89** m². In parallel flow (ends 20 and 3.3 K) $\Delta T_{lm} = 9.3$ K and $A = 3.59$ m², 25% more.

Worked example 2 – rating with fouling by ε–NTU. Buy a 3.0 m² unit. Clean: $C_{min} = 8360$ W/K, $C_r = 0.667$, $NTU = 0.897$, $\varepsilon =$ **0.511**, $Q = 0.511 \times 8360 \times 20 =$ **85.5** kW, loop outlet 29.8 °C – it meets the duty. After a year of scale, **fouling resistance** $R_f = 0.0002$ m²K/W gives $1/U = 1/2500 + 0.0002$, $U = 1667$: $NTU = 0.598$, $\varepsilon = 0.398$, $Q =$ **66.6** kW, outlet 32.0 °C – short. Sizing for the fouled U needs 4.33 m². More plates also mean more channels in parallel, lower velocity, lower pressure drop – and lower h: area, fouling and pump power trade against each other.

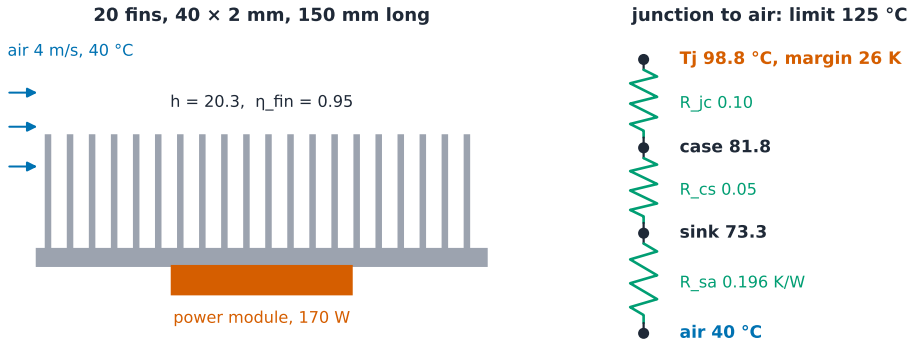
Common mistakes. Using the arithmetic mean temperature difference. Sizing on clean U. Swapping C_{min} and C_{max}.

Summary. $Q = UA\Delta T_{lm}$; counterflow needs least area; $\varepsilon = Q/C_{min}\Delta T_{max}$ from NTU and C_r; size on fouled U.

Thermal design

Why it matters. The pump motor's variable-speed drive loses about 3% of 5.65 kW – 170 W – in its power module. If the silicon passes 125 °C it fails. The cooling must be designed, with a margin that can be defended.

The idea, with a picture. Combine everything into one network: junction → case → sink → air. The sink's resistance comes from **fins** with convection set by the airflow (unit 10.3). A fin is less useful far from its base, which the **fin efficiency** η_f measures.



A finned heat sink on the drive's power module, and the resistance chain from junction to air

The derivation (fin efficiency). A thin fin conducts along its height while losing heat from both faces; the balance gives $d^2\theta/dx^2 = m^2\theta$ with $m = \sqrt{2h/(kt)}$ for a plate fin of thickness t. With an insulated tip, $\eta_f = \tanh(mH)/(mH)$. The sink's conductance is then $hA_{eff} = h(A_b + \eta_f A_f)$, and $R_{sa} = 1/(hA_{eff})$.

Worked example 1 – the allowed resistance. $T_j \le 125$ °C, air at 40 °C (cabinet worst case), $P = 170$ W, module $R_{jc} = 0.10$, interface $R_{cs} = 0.05$ K/W: $R_{sa,max} = (125 - 40)/170 - 0.15 = \mathbf{0.35}$ K/W.

Worked example 2 – the fin array. Aluminium ($k = 200$), base 200 × 150 mm, 20 fins 40 mm high and 2 mm thick, air 4 m/s along the 150 mm fins, so $h = 20.3$ from the plate correlation in unit 10.3: $m = \sqrt{2 \times 20.3/(200 \times 0.002)} = 10.1$ m^{-1}, $mH = 0.40$, $\eta_f = \mathbf{0.95}$. $A_f = 0.24$ m^2, $A_b = 0.024$ m^2: $hA_{eff} = 20.3(0.024 + 0.95 \times 0.24) = 5.10$ W/K, $R_{sa} = \mathbf{0.196}$ K/W. $T_j = 40 + 170(0.10 + 0.05 + 0.196) = \mathbf{98.8}$ °C – a **26 K** margin. Thirty fins would give 88.6 °C; 60 mm fins 89.8 °C.

The report. Heat balance closed: 170 W in, 170 W to air (radiation from the anodised sink, about 10% extra, ignored – a conservative assumption). Margins and assumptions: h from a flat-plate correlation for channel flow between fins (uncertain by ±25%: at $h = 15$, T_j rises to 110 °C, still below 125); fan failure is not covered and needs a thermal trip.

Common mistakes. Counting fin area at full efficiency. Using room temperature instead of the worst cabinet temperature. Reporting one T_j without the sensitivity to h.

Summary. One network from junction to air; $R_{sa} = 1/h(A_b + \eta_f A_f)$, $\eta_f = \tanh mH/mH$; state the margin, the weakest assumption and what happens if it is wrong.