

Topic 03

University Mechanical Engineering

What this course is, and how to use it

This is the third course in University Mechanical Engineering: **dynamics** – the study of motion and the forces that cause it. Course 1 (statics) drew free-body diagrams and set sums to zero; course 2 gave the mathematical instruments; this course sets those sums equal to mass times acceleration, and to their energy and momentum equivalents.

The running case continues: **the festival till** – 2.5 kg, a box 0.30 m tall and 0.20 m wide – being certified in its protective mount on the dynamics test floor. It is dropped, slid, spun, vibrated, impacted and rolled, and every number in this handout was computed for this course: the trajectories close, the Newton and energy routes give the same speed to four decimals, the impact chains conserve momentum to the digit, and the flywheel's energy ties to its spin-up work.

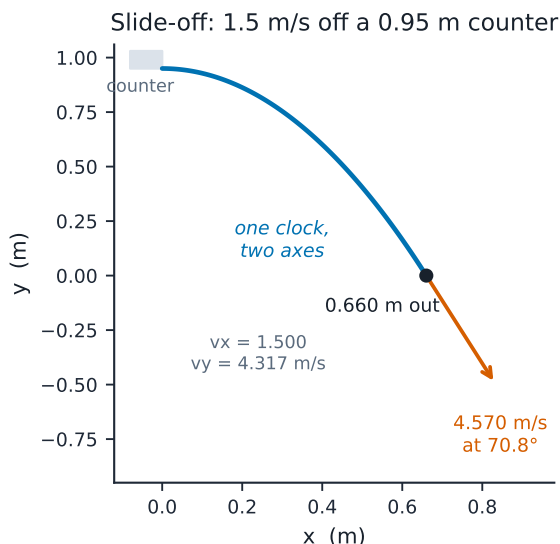
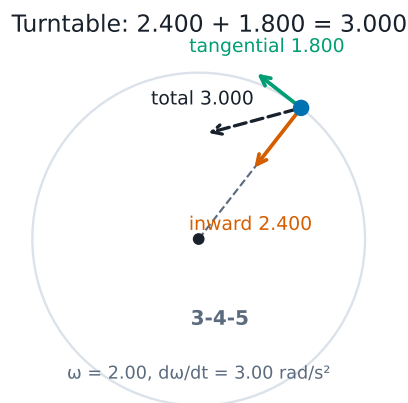
- Units 3.1–3.4 are the particle story (kinematics, forces, energy, momentum); 3.5–3.6 make the body rigid.

Each unit: the question, the machinery, two escalating worked examples checked a second way, the common mistakes, a summary.

Particle kinematics

Why it matters. Before any force can be computed, certification needs position, velocity and acceleration from the motion itself – the drop test's landing speed, the slide-off's landing point, the turntable's pull. Kinematics is reading the trace.

The idea, with a picture. The till slides off a counter 0.95 m high at 1.5 m/s horizontal: gravity owns the vertical axis, nothing owns the horizontal one, and the two share one clock. On the turntable, the same reading in polar components: inward and tangential.



*One clock, two axes - and in polar, one inward pull plus one tangential push

The definitions. A **particle** is a body whose size stops mattering – the till in free fall is one. The **position vector** $\mathbf{r}(t)$ differentiates once to **velocity** $\mathbf{v} = \dot{\mathbf{r}}$ and twice to **acceleration** $\mathbf{a} = \dot{\mathbf{v}}$; the path it traces is the **trajectory**. **Projectile motion** is the case where $a_x = 0$, $a_y = -g$: the two components share the same time t . In **polar coordinates**, with outward radial direction positive, $a_r = \ddot{r} - r\omega^2$ and $a_\theta = r\dot{\omega} + 2\dot{r}\omega$. For the fixed-radius turntable below, the inward acceleration magnitude is $r\omega^2$ and the tangential component is $r\dot{\omega}$, where **angular velocity** ω is in rad/s – never rev/s.

The derivation. The drop test, 1.80 m from rest: $v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 1.80} = \mathbf{5.943}$ m/s in $t = \sqrt{2h/g} = \mathbf{0.6058}$ s ✓ (the energy route of unit 3.3 will return this number). The slide-off: $t = \sqrt{2 \times 0.95/9.81} = \mathbf{0.4401}$ s, range = $1.5 \times 0.4401 = \mathbf{0.6601}$ m, landing speed $\sqrt{1.5^2 + 4.317^2} = \mathbf{4.570}$ m/s at $\arctan(4.317/1.5) = \mathbf{70.8^\circ}$ ✓. The turntable: $r = 0.600$ m at $\omega = 2.00$ rad/s speeding up at $\dot{\omega} = 3.00$ rad/s²: speed $v = \omega r = \mathbf{1.200}$ m/s, $a_r = \omega^2 r = \mathbf{2.400}$, $a_\theta = r\dot{\omega} = \mathbf{1.800}$, total $a = \sqrt{2.400^2 + 1.800^2} = \mathbf{3.000}$ m/s² ✓ – a 3-4-5 triangle drawn by the physics.

Worked example 1 – read the landing, both numbers. A report quotes “impact at 4.6 m/s” for the slide-off test. Which number is that, and is it right? It is $|\mathbf{v}|$ at landing: $\sqrt{v_x^2 + v_y^2}$ with $v_y = gt = 9.81 \times 0.4401 = 4.317$ gives 4.570 m/s – the report rounded up and lost the angle. The certification form wants the pair: 4.570 m/s at 70.8° below horizontal, because the mount’s arrest load (unit 3.3) and its shear load split along exactly those components.

Worked example 2 – escalate to the spinning table. The turntable test runs at $\omega = 2.00 \text{ rad/s}$ with $\dot{\omega} = 3.00 \text{ rad/s}^2$. Total acceleration 3.000 m/s^2 inward-and-forward at $\arctan(1.800/2.400) = 36.9^\circ$ ahead of the radius. The question the next unit will close: friction must supply $ma = 2.5 \times 3.000 = 7.50 \text{ N}$ – hold that number.

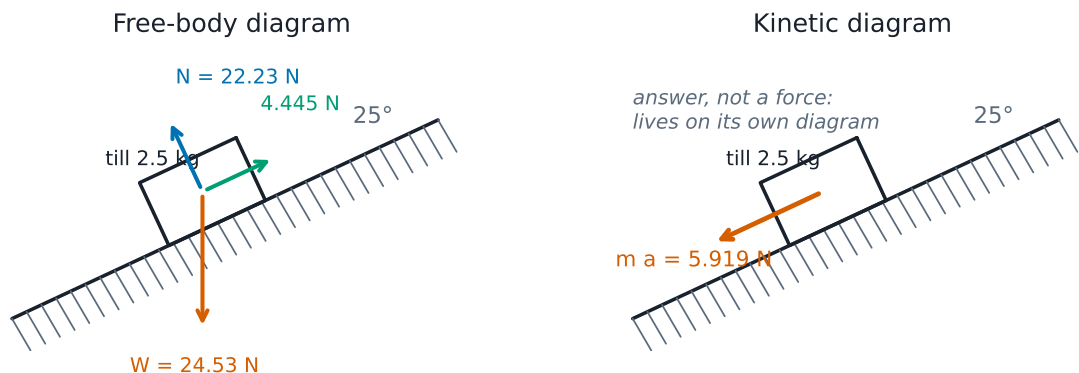
Common mistakes. Treating the two projectile components as independent clocks (there is one t); using ω in revolutions per second inside $a_r = \omega^2 r$ (convert: $\text{rev/s} \times 2\pi$); assuming speed is constant on a curve (the turntable's *speed* is constant while its *velocity* is not – direction is motion too).

Summary. Differentiate the trace; one clock for all components. Projectile: $a_y = -g$, $a_x = 0$. Polar: $a_r = \omega^2 r$ inward, $a_\theta = r\dot{\omega}$. The turntable's $2.400 + 1.800$ makes 3.000 at 36.9° .

Particle kinetics

Why it matters. The mount slides or holds, slides or tips, and the certification verdict is written by Newton's second law applied to a correctly drawn body – not by intuition.

The idea, with a picture. The till on the certification ramp, 25° to the horizontal: weight, **normal force** and **friction** drawn on the free-body diagram, and the single resultant ma drawn beside it as the kinetic diagram.



*Forces on one diagram, mass-times-acceleration on the other - never both on the same one

The definitions. The **free-body diagram** isolates one body and draws every external force on it. The **equation of motion** is $\sum \mathbf{F} = m\mathbf{a}$: forces on one side, the kinetic diagram on the other. **Friction** is $\mu_s N$ at most while static, exactly $\mu_k N$ once sliding; the **normal force** comes from the perpendicular force equation. On the simple incline here, with no other perpendicular force or normal acceleration, it is $mg \cos \beta$.

The derivation. The ramp, $\beta = 25^\circ$, $\mu_k = 0.20$: along the slope $mg \sin \beta - \mu_k mg \cos \beta = ma$, so $a = 9.81(0.4226 - 0.20 \times 0.9063) = \mathbf{2.368 \text{ m/s}^2}$; perpendicu-

lar, $N = mg \cos \beta = \mathbf{22.23}$ N, friction $= \mu_k N = \mathbf{4.445}$ N; from rest over 1.50 m, $v = \sqrt{2 \times 2.368 \times 1.50} = \mathbf{2.665}$ m/s ✓ (unit 3.3 re-derives this exact speed from energy). The accelerating cart: the till starts to slide back when the cart's acceleration exceeds $a_{\text{slip}} = \mu_s g = 0.35 \times 9.81 = \mathbf{3.434}$ m/s². The turntable demand: friction must give 7.50 N (unit 3.1), so $\mu \geq 7.50 / (2.5 \times 9.81) = \mathbf{0.3058}$.

Worked example 1 – the cart test. The shaker's transport cart brakes at 3.0 m/s². Does the till slide? Threshold 3.434 m/s² – no, with margin $(3.434 - 3.0) / 3.434 = \mathbf{12.6\%}$. The professional sentence names both numbers: the test pulls 3.0, the pair allows 3.43.

Worked example 2 – escalate to the spinning table's verdict. The turntable needs $\mu \geq 0.3058$; the mount's measured static coefficient is 0.35. It holds – margin $(0.35 - 0.3058) / 0.35 = \mathbf{12.6\%}$, the same ratio because both limits scale with the normal force. The certification writes: holds at 2.00 rad/s with $\dot{\omega} = 3.00$; first slip at $\dot{\omega} = 4.09$ rad/s², when $r\sqrt{\omega^4 + \dot{\omega}^2} = \mu_s g$.

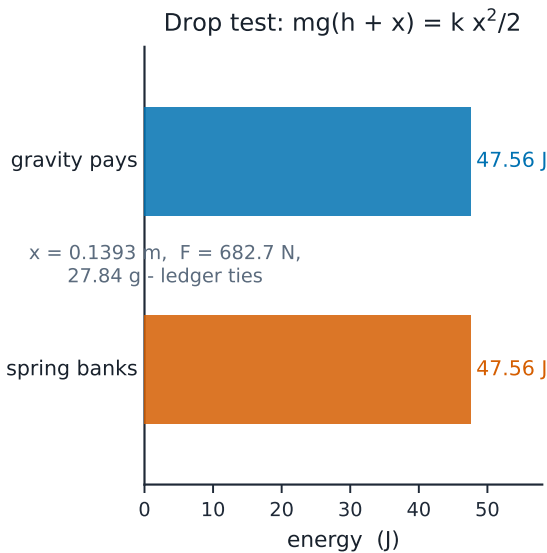
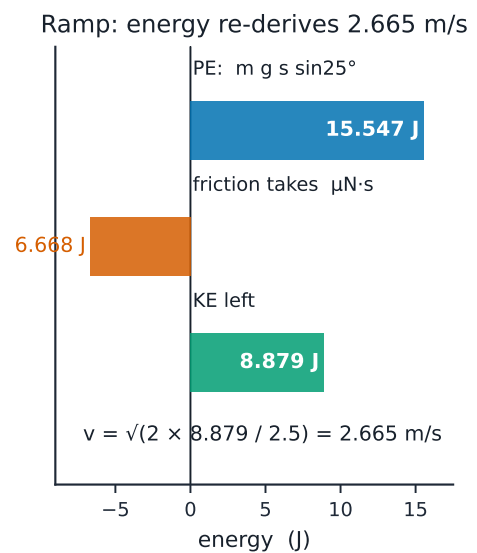
Common mistakes. Drawing ma as a force on the free-body diagram (it is the answer, on its own diagram); using μ_s after sliding starts; setting $N = mg$ on an incline; checking slide without checking tip (unit 3.6 closes that loop at 6.540 m/s²).

Summary. Free-body diagram, then axes along and across the motion. N from the perpendicular equation, friction μN with the right μ , then integrate. Ramp: $a = 2.368$, $v = 2.665$ m/s; cart slip 3.434; turntable needs μ 0.3058.

Work and energy

Why it matters. The mount's spring force varies with position – integrating Newton every time is possible once and pointless afterwards. Energy does the integral once and bookkeeps it forever.

The idea, with a picture. The drop test onto the mount as a spring: gravity pays in joules for the whole fall *plus* the deflection; the spring banks every joule at maximum compression.



*The ledger must balance to the last joule - that is the second-way check

The definitions. **Work** is $U = \int \mathbf{F} \cdot d\mathbf{r}$. Work done by an ideal spring is the decrease in its elastic potential energy; it can be positive or negative. **Kinetic energy**

$T = \frac{1}{2}mv^2$ is a scalar: no components, no direction. **Potential energy** is work already banked by position: mgh for gravity, $\frac{1}{2}kx^2$ for a spring of **spring constant** k . The **energy method** is $T_1 + V_1 + U_{1 \rightarrow 2} = T_2 + V_2$: one equation for how-fast and how-far questions, blind to time.

The derivation. The mount, $k = 4900$ N/m, dropped from 1.80 m: gravity supplies $mg(h+x)$ while the spring stores $\frac{1}{2}kx^2$ - the $+x$ is the deflection gravity keeps working through, the term every rushed report drops. $2450x^2 = 24.525(1.80 + x)$ gives $x_{max} = 0.1393$ m, force $kx = 682.7$ N, deceleration 273.1 m/s² = **27.84 g** ✓ - and both ledger sides read **47.56 J** ✓. The ramp, again: $mgs \sin \beta - \mu_k mg \cos \beta s = \frac{1}{2}mv^2$: $15.547 - 6.668 = 8.879$ J, $v = \sqrt{2 \times 8.879 / 2.5} = 2.665$ m/s ✓ - the same speed Newton gave in unit 3.2, by a second route.

Worked example 1 – the arrest load without integrating. What is the maximum force on the till? Energy: $x_{max} = 0.1393$ m, $F_{max} = kx_{max} = 682.7$ N - no force integral was ever evaluated. The average spring force is $\frac{1}{2}F_{max} = 341.4$ N, which is why energy answers and intuition checks: the peak is twice the average.

Worked example 2 – escalate to the rating. The till's electronics are rated to 30 g. The drop test peaks at 27.84 g - it passes with margin $(30 - 27.84)/30 = 7.2\%$. But halve the drop height and the peak does not halve: at $h = 0.900$ m the same equation gives $x = 0.1001$ m, $F = 490.3$ N, 20.0 g - two thirds of the rating for half the height, because gravity keeps paying through the deflection.

Common mistakes. Dropping the $+x$ in $mg(h+x)$ (the most common error in drop-test reports); double-counting gravity as both a force and a potential; friction

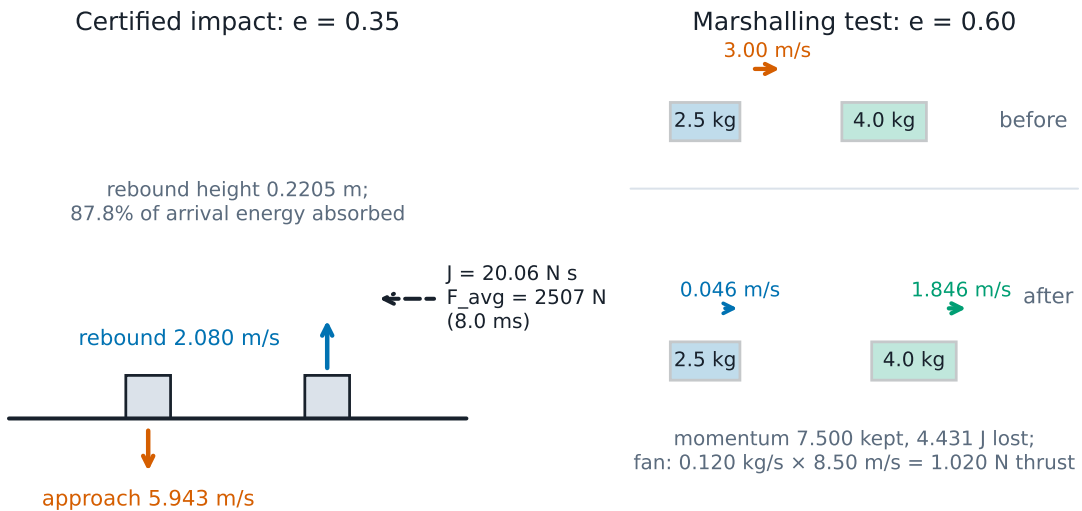
work with the wrong sign (it is $-\mu_k N s$ against the motion); asking energy for a *time* – that is unit 3.2's integration.

Summary. $T_1 + V_1 + U_{1 \rightarrow 2} = T_2 + V_2$, with springs at $\frac{1}{2} k x^2$. Drop test: $x = 0.1393$ m, 682.7 N, 27.84 g, ledger 47.56 J = 47.56 J. Ramp: $15.547 - 6.668 = 8.879$ J $\rightarrow$ 2.665 m/s.

Impulse and momentum

Why it matters. The certified impact lasts 8 milliseconds – too fast to integrate forces, exactly right for their time integral. Impulse and momentum are how impacts are specified, tested and survived.

The idea, with a picture. The till meets the certified floor at 5.943 m/s and leaves at 2.080 m/s; the impulse is the whole story between the two pictures.



*Two frozen pictures and one arrow of impulse between them - that is an impact analysis

The definitions. **Linear momentum** is $\mathbf{p} = m\mathbf{v}$ – a vector. **Impulse** is $\mathbf{J} = \int \mathbf{F} dt = \Delta\mathbf{p}$: forces may be unknown and wild, their time integral is not. The **coefficient of restitution** e is the separation speed over the approach speed at the contact point – 1 for elastic, 0 for dead. **Steady flow** has time-independent conditions at each fixed point. For a steady single-inlet/single-outlet stream, net external force on the fluid is $\dot{m}(\mathbf{v}_{out} - \mathbf{v}_{in})$. This includes pressure, weight and wall forces. The fluid force on a wall opposes that wall's force on the fluid.

The derivation. The drop impact at $v = 5.943$ m/s with $e = 0.35$: rebound $ev = 2.080$ m/s, impulse $J = m(v + ev) = 2.5 \times 1.35 \times 5.943 = 20.06$ N · s, rebound height $v_r^2/2g = 0.2205$ m ✓. The marshalling test: the 2.5 kg cart at 3.00 m/s into a 4.0 kg cart at rest, $e = 0.60$: momentum $7.50 = 2.5v_1 + 4.0v_2$ and separation $v_2 - v_1 = 0.60 \times 3.00$ give $v_1 = 0.046$, $v_2 = 1.846$ m/s ✓ – momentum checks to the

digit ($2.5 \times 0.046 + 4.0 \times 1.846 = 7.50$), kinetic energy does not: $11.25 \rightarrow 6.82$ J, **4.431** J to dents and heat ✓. The fan: $\dot{m} = 0.120$ kg/s out at 8.50 m/s gives thrust $\dot{m}v = 1.020$ N ✓.

Worked example 1 – the average force. The certified floor contact lasts $\Delta t = 8.0$ ms. Average force $= J/\Delta t = 20.06/0.0080 = \mathbf{2507}$ N – a number the mount’s welds are sized against, obtained without ever knowing the force curve. Gravity’s impulse over the same 8 ms is $mg\Delta t = 0.196$ N · s, 1.0% of J : the short-impact licence to conserve momentum is now quantified, not assumed.

Worked example 2 – escalate to the loss budget. How much energy may the arrest structure absorb elsewhere? The impact’s kinetic budget: approach $\frac{1}{2}mv^2 = 44.15$ J, rebound $\frac{1}{2}mv_r^2 = 5.41$ J, so the floor and mount dissipate **38.74 J** – 87.8% of the incoming energy, which is what an e of 0.35 means and what the certification’s crush-zone specification buys.

Common mistakes. Losing the vector sign (down-in and up-out momenta subtract – $J = m(v + ev)$, not $m(v - ev)$); applying e to velocities instead of the *separation/approach* pair; conserving kinetic energy because momentum is conserved (only $e = 1$ does that); drawing the system boundary so wide that external impulses sneak back in.

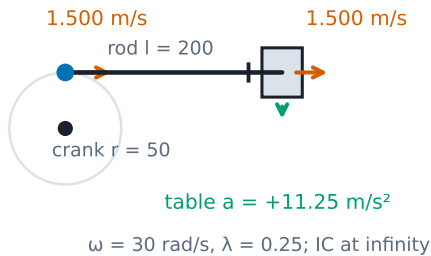
Summary. $\sum \mathbf{J} = \Delta \mathbf{p}$; e = separation over approach. Impact: $J = 20.06$ N · s, average force 2507 N over 8 ms. Carts: 0.046 and 1.846 m/s, 4.431 J lost. Fan thrust $\dot{m}v = 1.020$ N.

Rigid-body kinematics

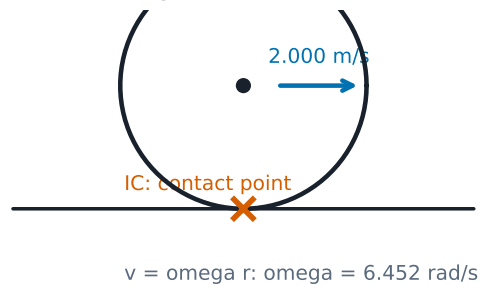
Why it matters. The shaker’s drive is a mechanism, not a particle: crank, rod, table, each point moving differently, joined by geometry. Rigid-body kinematics is constraint before differentiation.

The idea, with a picture. The shaker’s slider-crank at the quarter-turn position, and the transport wheel rolling without slip – two machines, one method: find the constraint, then differentiate.

Slider-crank at 90°: pure translation



4.000 m/s
Rolling: IC at the contact



*One mechanism, two velocities, one zero-velocity point - the geometry does the differentiating

The definitions. A **rigid body** keeps every pair of its points at fixed distance. The **slider-crank mechanism** converts rotation to reciprocation: crank r , rod l , ratio $\lambda = r/l$. The **instant centre** of any body planar motion is the point of zero velocity at that instant – every other point turns about it *now*. **Rolling without slipping** ties the wheel to the ground: $v = \omega r$, contact point at rest; in a **rotating frame** the same motion is described with time-varying basis vectors.

The derivation. The shaker drive, $r = 50 \text{ mm}$, $l = 200 \text{ mm}$, $\omega = 30 \text{ rad/s}$: piston position $x = r \cos \theta + \sqrt{l^2 - r^2 \sin^2 \theta}$. At $\theta = 90^\circ$: $x = \sqrt{l^2 - r^2} = \mathbf{0.1936 \text{ m}}$, and differentiating, $\dot{x} = -r\omega = \mathbf{-1.500 \text{ m/s}}$ while $\ddot{x} = +r\omega^2\lambda = \mathbf{+11.25 \text{ m/s}^2}$ ✓; at $\theta = 0^\circ$ the acceleration peaks at $\ddot{x} = -r\omega^2(1 + \lambda) = \mathbf{-56.25 \text{ m/s}^2}$ ✓ – the same crank swings its table through a 5:1 acceleration range. The wheel, $r = 0.310 \text{ m}$ rolling at 2.00 m/s : $\omega = v/r = \mathbf{6.452 \text{ rad/s}}$, contact point at rest – the **instant centre** – centre at 2.00 , top at $\mathbf{4.000 \text{ m/s}}$ ✓.

Worked example 1 – the rod is translating at quarter-turn. At $\theta = 90^\circ$ the crank pin's velocity is horizontal (perpendicular to the vertical crank) and the piston's velocity is horizontal by its slide – both perpendiculars to the rod's direction are parallel, so the rod's instant centre is at infinity: pure translation, every rod point at 1.500 m/s ✓. The check that never fails: compute the rod's end-point speeds independently – both 1.500 m/s , same direction.

Worked example 2 – escalate to the fatigue spec. The table's acceleration cycles between $+11.25$ at quarter-turn and -56.25 m/s^2 at dead centre each revolution – a 5:1 load range at $30 \text{ rad/s} = 4.77 \text{ Hz}$. The mount's fatigue file is written from exactly this kinematic output: 2.9×10^6 cycles per week at the peak, the number the weld sizing (unit 3.6) consumes.

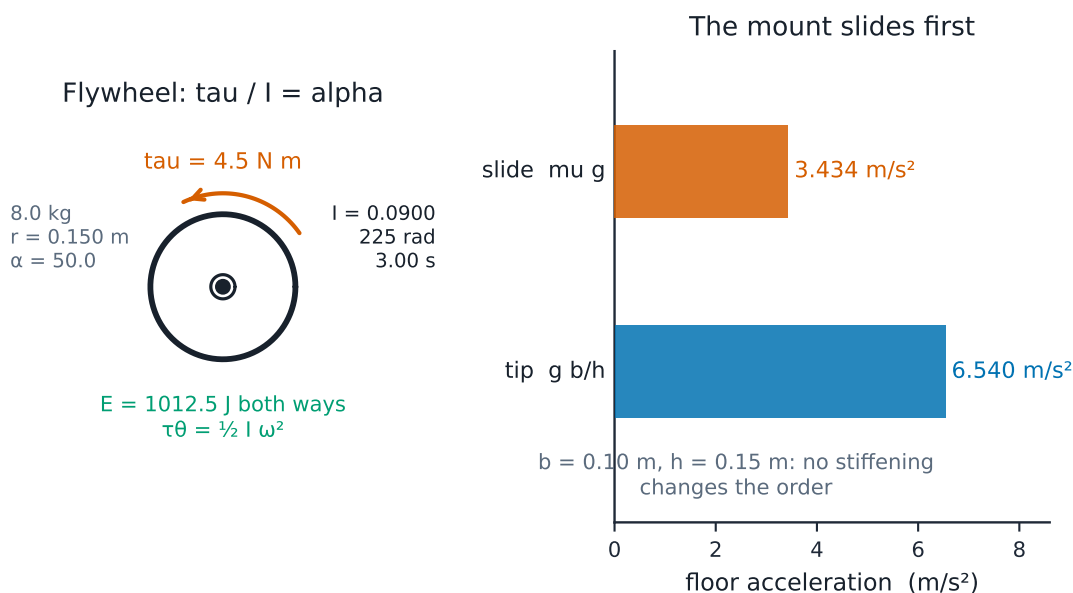
Common mistakes. Feeding rev/s into $a = \omega^2 r$; believing the instant centre is a fixed pivot (it travels with the body's configuration); forgetting the wheel's top moves at $2v$ while the contact sleeps; differentiating a constraint without stating the frame.

Summary. Constraint geometry first, differentiation second. Slider-crank at 90° : $v = r\omega = 1.500$ m/s, $a = r\omega^2\lambda = 11.25$ m/s²; extremes 5:1. Rolling: $v = \omega r$, IC at contact, top at $2v$.

Rigid-body kinetics

Why it matters. Spin stores energy and resists torque by geometry, not just mass; the certification file's last rows are the flywheel's inertia and the mount's tip-versus-slide verdict.

The idea, with a picture. The spin-test flywheel – a disk with a torque and a number I – beside the two failure modes of the mount on a rough accelerating floor: slide at 3.434 , tip at 6.540 m/s².



*Mass resists push, inertia resists spin - and a box meets its smaller limit first

The definitions. The **mass moment of inertia** $I = \int r^2 dm$ is the spin analogue of mass – a disk about its axle carries $\frac{1}{2}mr^2$. **Torque** τ turns it: for rotation about a fixed axis with constant I , $\sum \tau = I\alpha$ with **angular acceleration** α in rad/s². The axial **angular momentum** is $L = I\omega$ for this fixed-axis model. Total angular momentum about a fixed inertial point is conserved when the net external torque about that point is zero. **Tipping** begins when the resultant force crosses the base edge: for a box of half-width b and centre-of-mass height h , at $a = gb/h$.

The derivation. The flywheel, $m = 8.0$ kg, $r = 0.150$ m, $\tau = 4.5$ N · m: $I = \frac{1}{2}mr^2 =$

0.0900 kg · m², $\alpha = \tau/I = \mathbf{50.00}$ rad/s², so 3.00 s of spin-up reaches $\omega = \mathbf{150}$ rad/s through $\theta = \frac{1}{2}\alpha t^2 = \mathbf{225}$ rad = **35.81** revolutions ✓, storing $\frac{1}{2}I\omega^2 = \mathbf{1012.5}$ J ✓ – equal to the torque’s work $\tau\theta = 4.5 \times 225 = 1012.5$ J ✓, the energy ledger again. The rolling disk down 15°: $a = g \sin \beta / (1 + I/mr^2) = \frac{2}{3} \times 9.81 \times 0.2588 = \mathbf{1.693}$ m/s². The mount’s verdict: tip at $a_{tip} = g \times 0.10/0.15 = \mathbf{6.540}$ m/s², slide at $\mu_s g = \mathbf{3.434}$ – it **slides first**, and no stiffening changes that order.

Worked example 1 – measure I by rundown. With the drive off, the flywheel coasts from 150 rad/s to rest in 40.0 s: $\alpha = -3.75$ rad/s², so bearing friction is $\tau_f = I|\alpha| = \mathbf{0.3375}$ N · m – the inertia measured from a stopwatch, no geometry taken on faith. The energy cross-check: friction dissipates $\frac{1}{2}I\omega^2 = 1012.5$ J over $\theta = \omega^2/2|\alpha| = 3000$ rad, and $0.3375 \times 3000 = 1012.5$ J ✓.

Worked example 2 – escalate to the rolling split. A 5.0 kg disk rolls 1.50 m down the 15° ramp: $a = 1.693$ m/s² gives $v = \sqrt{2 \times 1.693 \times 1.50} = \mathbf{2.253}$ m/s. The energy split: translational $\frac{1}{2}mv^2 = \mathbf{12.70}$ J, rotational $\frac{1}{2}I\omega^2 = \frac{1}{4}mv^2 = \mathbf{6.35}$ J, total **19.04** J – and the drop was $mg \times 1.50 \sin 15^\circ = 19.04$ J ✓. Two thirds translation, one third rotation: the disk’s own geometry taxes every ramp.

Common mistakes. Quoting I without its axis (about the axle and about the edge differ by mr^2); using the tipping formula with full width instead of half-width, or rim height instead of centre-of-mass height; adding rolling energies without the $(1 + I/mr^2)$ factor; treating $\sum \tau = I\alpha$ as valid about any point (a fixed axle or the centre of mass, or pay with extra terms).

Summary. $\sum \tau = I\alpha$, disk $I = \frac{1}{2}mr^2$: flywheel 0.0900, 50.00 rad/s², 1012.5 J both ways. Rolling disk $a = \frac{2}{3}g \sin \beta = 1.693$, energy split 2:1. The mount slides at 3.434 before it tips at 6.540.

The project sheet (reproducible)

The certification file’s dynamics page, one page:

- Kinematics: drop from 1.80 m lands at 5.943 m/s in 0.6058 s; slide-off lands at 4.570 m/s, 70.8°, 0.6601 m out; turntable $2.400 + 1.800 = 3.000$ m/s².
- Newton: ramp $a = 2.368$ m/s², $v = 2.665$ m/s, $N = 22.23$ N; cart slip at 3.434; turntable needs μ 0.3058.
- Energy: mount compresses 0.1393 m, force 682.7 N, 27.84 g, ledger 47.56 J = 47.56 J; ramp $15.547 - 6.668 = 8.879$ J $\rightarrow$ 2.665 m/s.
- Momentum: impulse 20.06 N · s, average force 2507 N (8 ms), rebound 2.080 m/s; carts $0.046 + 1.846$ m/s, 4.431 J lost; fan thrust 1.020 N.
- Mechanism: quarter-turn $v = 1.500$ m/s, $a = 11.25$ m/s², extremes 5:1; wheel $\omega = 6.452$ rad/s, top 4.000 m/s.

- Rigid body: flywheel $I = 0.0900$, $\alpha = 50.00$, $225 \text{ rad} = 35.81 \text{ rev}$, 1012.5 J ; rolling disk 1.693 m/s^2 , split $12.70 + 6.35 = 19.04 \text{ J}$; mount slides at 3.434 before tipping at 6.540 m/s^2 .

Recompute any line from $g = 9.81$, $m = 2.5 \text{ kg}$ and the dimensions above – a reader with a calculator can check every number, and the machine suite that generated them runs in under a second.