2.4 Numerical methods

Name: _____ Class: _____ Date: _____ **Total: 14 marks**

KEY WORDS truncation error • round-off error

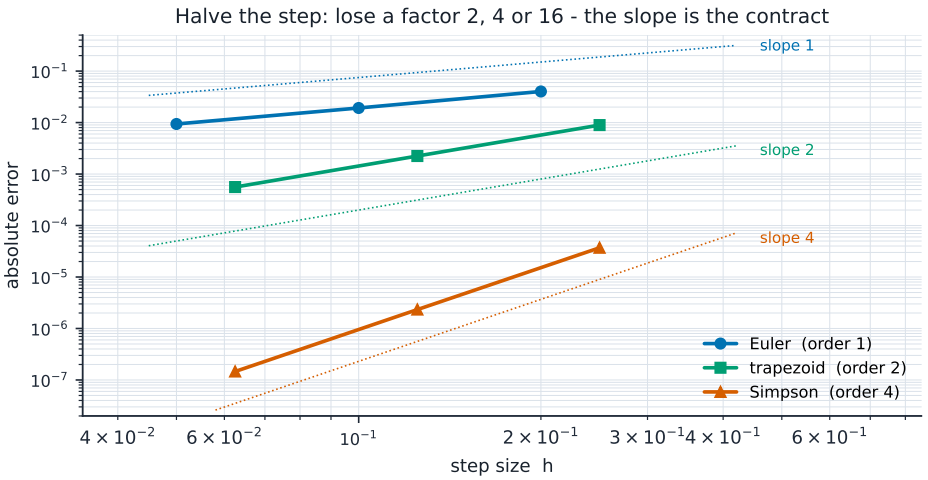
Objective

Measure convergence orders from a run, choose a step size from a budget, and know where round-off takes over.

1 Key knowledge and worked examples

■ The contract of slopes

Euler O(h); trapezoid O(h^2); Simpson O(h^4). Error estimate with every result; validate on a closed form first.



Halve the step: lose a factor 2, 4 or 16

■ Worked example. Measured runs

Euler 4.0e-2 to 9.4e-3; trapezoid 1.04e-2 to 6.5e-4; Simpson 2.4e-5 to 2.4e-9 - the orders confirmed by halving ✓.

2 Practice

The integral $I = \int_0^1 e^x\,dx = e - 1 = 1.718282$ is approximated by the trapezoidal rule and Simpson's rule. Measured absolute errors: trapezoid 8.94×10^{-3} at $n = 4$, 2.24×10^{-3} at $n = 8$; Simpson 3.70×10^{-5} at $n = 4$, 2.33×10^{-6} at $n = 8$.

2.1 From the two trapezoid errors, the order, shown as a ratio. [2]

2.2 The predicted trapezoid error at $n = 16$ (two significant figures), and the Simpson order from its own pair. [4]

2.3 A budget of 10^{-6}: the n Simpson needs (power of two), and the n the trapezoid would need to the nearest power of ten. [4]

2.4 Why does the Simpson error not keep falling forever? Name the mechanism and its per-step scale. [4]

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Solutions

2.1

- $8.94/2.24 = 3.99 \approx 4$ per halving: order 2.

2.2

- Trapezoid at 16: $2.24 \times 10^{-3}/4 = 5.6 \times 10^{-4}$.
- Simpson $3.70 \times 10^{-5} \to 2.33 \times 10^{-6}$: ratio $15.9 \approx 16$: order 4.

2.3

- Simpson from $n = 8$ needs $2.33 \times 10^{-6}/16 < 10^{-6}$: $n = 16$ suffices.
- Trapezoid from $n = 8$: $2.24 \times 10^{-3}/4^p < 10^{-6}$ needs $p \approx 5.6$ halvings - n near $2^{14} \approx 16{,}000$.

2.4

- Round-off: each panel adds floating-point noise of order 10^{-16}.
- Past $\sim 10^5$ panels the accumulated round-off ($\sim 10^{-10}$) exceeds truncation - the U-curve floor.

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