12.3 Transfer functions and response

Name: _____ Class: _____ Date: _____ **Total: 18 marks**

KEY WORDS poles • zeros • stable • overshoot • crossover • phase margin

Objective

Read a second-order system's response from its poles, find overshoot, peak and settling times, design a proportional gain for a damping ratio, and find the phase margin.

1 Key knowledge and worked examples

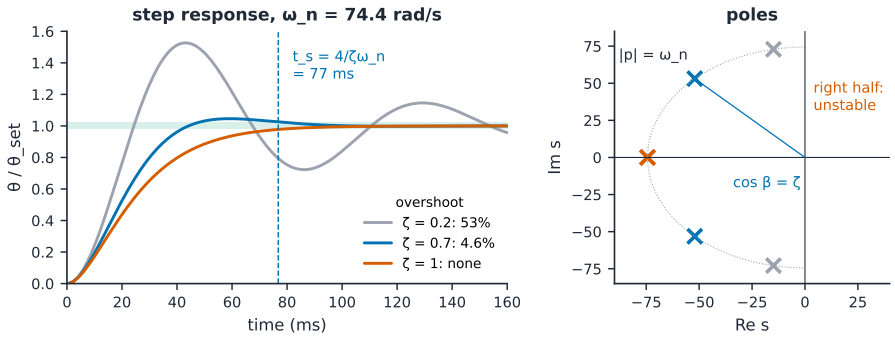
■ Poles set the response

- $\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$ has **poles** $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$; **zeros** are the numerator's roots.
- All poles in the left half-plane: **stable** .
- **Overshoot** $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$; peak at $t_p = \pi/\omega_d$; settling (2%) $t_s \approx 4/(\zeta\omega_n)$.
- Open loop $L(j\omega)$: $|L| = 1$ at the **crossover** ω_c; **phase margin** $= 180^\circ + \angle L(j\omega_c)$, $\zeta \approx PM/100$.

■ Worked example. The position loop with P control

Plant $G = 0.398/[s(0.0096s+1)]$; closed loop $0.398K_p/(0.0096s^2+s+0.398K_p)$.

- For $\zeta = 0.7$: $0.398K_p = 1/(4\zeta^2 \times 0.0096) = 53.1$, so $K_p =$ **134** V/rad, $\omega_n = 74.4$ rad/s.
- $M_p =$ **4.6**%, $t_s = 77$ ms.
- $|L| = 1$ at $\omega_c = 48.2$ rad/s: $PM =$ **65.2**°.



2 Practice

2.1 A system has $\zeta = 0.5$ and $\omega_n = 10$ rad/s. Find its poles, overshoot (one decimal, %), peak time (three decimals, s) and settling time. [4]

2.2 For $G(s) = 20/(s^2+4s+20)$ find the poles, ω_n (two decimals) and ζ (three decimals). Is it stable? Would $1/[(s-1)(s+5)]$ be? [4]

3 Exam-style questions

3.1 A valve drive has $G = 0.414/[s(0.0155s + 1)]$ and P control. (a) Show the closed loop is $s^2 + 64.5s + 26.7K_p = 0$ (characteristic equation). (b) Find K_p for $\zeta = 0.7$ (one decimal, V/rad), ω_n, M_p, t_p and t_s (ms). (c) Find the crossover (one decimal, rad/s) and phase margin (one decimal, degrees), and check $\zeta \approx PM/100$. [10]

Solutions

2.1

- $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = \mathbf{-5 \pm 8.66j}$.
- $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} = e^{-0.5\pi/0.866} = \mathbf{16.3}\%$.
- $t_p = \pi/\omega_d = \pi/8.66 = \mathbf{0.363}$ s; $t_s = 4/(\zeta\omega_n) = \mathbf{0.8}$ s.

2.2

- $s^2 + 4s + 20 = 0$: $s = \mathbf{-2 \pm 4j}$.
- $\omega_n = \sqrt{20} = \mathbf{4.47}$ rad/s; $\zeta = 2/4.47 = \mathbf{0.447}$.
- Both poles in the left half-plane: stable.
- The second has a pole at $s = +1$, in the right half-plane: its response grows like e^t, so it is unstable.

3.1

- (a) $T = KG/(1 + KG)$ gives $0.0155s^2 + s + 0.414K_p = 0$; divide by 0.0155: $s^2 + 64.5s + 26.7K_p = 0$.
- (b) $2\zeta\omega_n = 64.4$, so $\omega_n = \mathbf{46.0}$ rad/s.
- $\omega_n^2 = 26.7K_p$: $K_p = 2116/26.67 = \mathbf{79.4}$ V/rad.
- $M_p = e^{-0.7\pi/\sqrt{0.51}} = \mathbf{4.6}\%$; $\omega_d = 32.9$ rad/s, $t_p = \pi/\omega_d = \mathbf{95.6}$ ms; $t_s = 4/(\zeta\omega_n) = 4/(0.7 \times 46.0) = \mathbf{124}$ ms.
- (c) $L = 32.9/[j\omega(0.0155j\omega+1)]$; $|L| = 1$ when $\omega\sqrt{1 + (0.0155\omega)^2} = 32.9$, so $\omega_c = \mathbf{29.8}$ rad/s.
- $PM = 180° - 90° - \tan^{-1}(0.0155 \times 29.8) = \mathbf{65.2}°$.
- $PM/100 = 0.65$, close to the 0.7 designed for.