UNIVERSITY MECHANICAL ENGINEERING • EXERCISE SHEET

12.1 Lumped-parameter modelling

Name: _____ Class: _____ Date: _____ **Total: 17 marks**

KEY WORDS lumped parameters • state-space form • transfer function • time constant

Objective

Model a tank, a mount and a circuit as lumped elements, find a first-order time constant and its step response, and write a second-order model in state-space form.

1 Key knowledge and worked examples

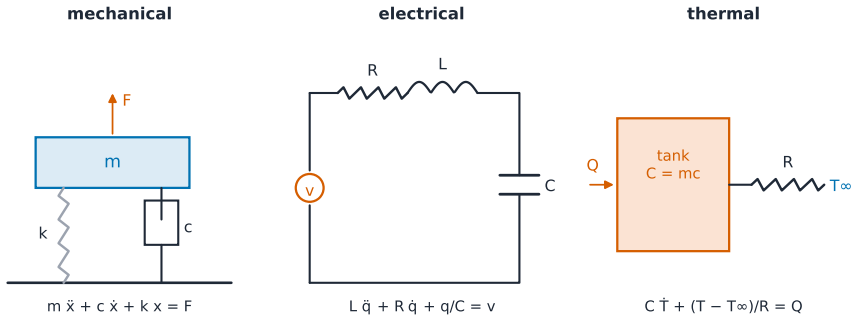
■ **One equation form for every domain**

- **Lumped parameters** : two stores and one dissipator per domain (mass, spring, damper; L, C, R; thermal C and R).
- **Transfer function** (zero initial conditions): $m\ddot{x} + c\dot{x} + kx = F$ gives $G(s) = 1/(ms^2 + cs + k)$.
- Heated tank: $C\dot{T} + (T - T_\infty)/R = Q$, $G(s) = R/(RCs + 1)$, **time constant** $\tau = RC$; step response $\Delta T = QR(1 - e^{-t/\tau})$.
- **State-space form** $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ with $x_1 = x$, $x_2 = \dot{x}$: $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$.

■ **Worked example. The loop's reservoir**

200 L of water, $UA = 100$ W/K to the room, 1 kW heater.

- $C = mc = 200 \times 4180 = \mathbf{836}$ kJ/K; $R = 1/UA = \mathbf{0.01}$ K/W.
- $\tau = RC = \mathbf{8360}$ s = 2.3 h.
- Final rise $QR = \mathbf{10}$ K; 63% of it after one τ.



2 Practice

2.1 A 150 L water tank ($c = 4180$ J/(kg · K)) loses heat through $UA = 50$ W/K. Find C (kJ/K), R (K/W) and τ (s and hours, two decimals). [3]

2.2 A 2 kW heater is switched on. Find the final temperature rise, the rise after 2 h (one decimal, K), and the time to reach a 30 K rise (two decimals, h). [4]

2.3 State one assumption that makes the tank a single lump. [1]

3 Exam-style questions

3.1 (a) A 50 kg mass on a spring of 20 kN/m with a 400 N · s/m damper is pushed by a force F. Write $\mathbf{A}$ and $\mathbf{B}$, find the eigenvalues of $\mathbf{A}$, and from them ω_n and ζ. (b) A series RLC circuit has $L = 0.5$ H, $R = 40\ \Omega$ and $C = 500$ µF. Write its equation in the charge q, name the mechanical analogue of each element, and find ω_n (two decimals, rad/s) and ζ (three decimals). [9]

Solutions

2.1

- $C = mc = 150 \times 4180 = \mathbf{627}$ kJ/K.
- $R = 1/UA = \mathbf{0.02}$ K/W.
- $\tau = RC = 0.02 \times 627\,000 = \mathbf{12\,540}$ s $= \mathbf{3.48}$ h.

2.2

- Final rise $= QR = 2000 \times 0.02 = \mathbf{40}$ K.
- $\Delta T = QR(1 - e^{-t/\tau}) = 40(1 - e^{-7200/12\,540}) = \mathbf{17.5}$ K.
- $30 = 40(1 - e^{-t/\tau})$ gives $t = \tau \ln 4 = \mathbf{4.83}$ h.

2.3

- The water is well mixed, so it has one temperature (or: UA is constant; the wall stores no heat).

3.1

- (a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -400 & -8 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$.
- $\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 + 8\lambda + 400 = 0$, so $\lambda = \mathbf{-4 \pm 19.6j}$.
- $\omega_n = |\lambda| = \sqrt{400} = \mathbf{20}$ rad/s; $\zeta = 4/20 = \mathbf{0.2}$.
- (b) $L\ddot{q} + R\dot{q} + q/C = V$: inductance $\leftrightarrow$ mass, resistance $\leftrightarrow$ damper, $1/C \leftrightarrow$ spring stiffness, voltage $\leftrightarrow$ force.
- $\omega_n = 1/\sqrt{LC} = 1/\sqrt{0.5 \times 500 \times 10^{-6}} = \mathbf{63.25}$ rad/s.
- $\zeta = (R/2)\sqrt{C/L} = 20\sqrt{0.001} = \mathbf{0.632}$.