11.6 Multi-degree-of-freedom systems

Name: _____ Class: _____ Date: _____ **Total: 17 marks**

KEY WORDS eigenvalue problem • mode shape • mode superposition

Objective

Write the mass and stiffness matrices of a two-mass system, find its natural frequencies and mode shapes, and find its forced response and floor force at the running speed.

1 Key knowledge and worked examples

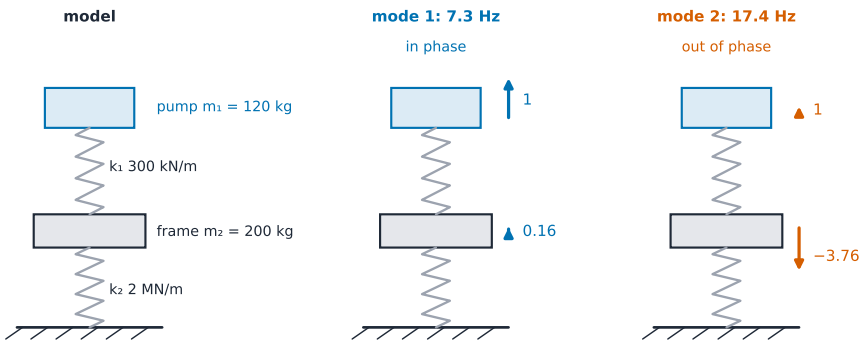
■ **Matrices, modes, superposition**

- $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}$; for a machine m_1 on mounts k_1 above a frame m_2 on feet k_2: $\mathbf{K} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix}$.
- **Eigenvalue problem** $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ gives the natural frequencies; each **mode shape** from row 1: $u_2/u_1 = 1 - \omega^2 m_1/k_1$.
- Modes are orthogonal, so the response is a sum of single-degree responses: **mode superposition** .
- Forced: solve $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{X} = \mathbf{F}_0$; floor force $k_2 X_2$.

■ **Worked example. The pump on its frame**

$m_1 = 120$ kg, $k_1 = 300$ kN/m, $m_2 = 200$ kg, $k_2 = 2$ MN/m.

- $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$: $\omega_1 = \mathbf{45.8}$ rad/s (7.30 Hz), $\mathbf{u}_1 = (1, 0.160)$, in phase.
- $\omega_2 = \mathbf{109.1}$ rad/s (17.4 Hz), $\mathbf{u}_2 = (1, -3.76)$, frame against pump.
- At 24 Hz with 27.3 N on the pump: pump 11.4 µm, frame 1.5 µm, floor force $k_2 X_2 = \mathbf{3.0}$ N.



2 Practice

A 100 kg fan sits on mounts of total stiffness $k_1 = 200$ kN/m on a 150 kg frame whose feet have $k_2 = 1.5$ MN/m.

2.1 Write **M** and **K**. [2]

2.2 Show that $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ becomes $15\,000\,\omega^4 - 2.0 \times 10^8\,\omega^2 + 3.0 \times 10^{11} = 0$, and find both natural frequencies (two decimals, rad/s and Hz). [5]

3 Exam-style questions

3.1 (a) Find both mode shapes as $(1, u_2)$ (three decimals) and describe each. (b) The fan's unbalance gives $F_0 = 49.3$ N on m_1 at 157.1 rad/s. Find the pump and frame amplitudes (one decimal and two decimals, µm) from $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{X} = (F_0, 0)$, and the floor force (one decimal, N). (c) Compare with treating the fan alone on k_1 above a rigid floor, and say what would happen if mode 2 moved to 25 Hz. [10]

Solutions

2.1

- $\mathbf{M} = \mathrm{diag}(100, 150)$ kg.
- $\mathbf{K} = \begin{bmatrix} 200 & -200 \\ -200 & 1700 \end{bmatrix}$ kN/m.

2.2

- $(k_1 - \omega^2 m_1)(k_1 + k_2 - \omega^2 m_2) - k_1^2 = 0$.
- Expanding: $m_1 m_2 \omega^4 - [m_1(k_1 + k_2) + m_2 k_1]\omega^2 + k_1 k_2 = 0$, that is $15\,000\omega^4 - 2.0 \times 10^8 \omega^2 + 3.0 \times 10^{11} = 0$.
- $\omega^2 = 1722.5$ or $11\,611$.
- $\omega_1 = \mathbf{41.50}$ rad/s (**6.61** Hz); $\omega_2 = \mathbf{107.75}$ rad/s (**17.15** Hz).

3.1

- (a) $u_2 = 1 - \omega^2 m_1 / k_1 = 1 - \omega^2/2000$.
- Mode 1: $u_2 = 1 - \omega_1^2 m_1/k_1 = 1 - 1722.5/2000 = \mathbf{0.139}$: both move together, the mounts flex.
- Mode 2: $u_2 = 1 - \omega_2^2 m_1/k_1 = 1 - 11\,611/2000 = \mathbf{-4.805}$: the frame bounces against the fan on its stiff feet.
- (b) $\omega^2 = 24\,674$: $\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} -2.267 & -0.2 \\ -0.2 & -2.001 \end{bmatrix} \times 10^6$ N/m.
- Solving: $X_1 = \mathbf{-22.0}$ µm, $X_2 = \mathbf{2.19}$ µm; the fan moves in antiphase with the force (above both modes).
- Floor force $= k_2 X_2 = 1.5 \times 10^6 \times 2.19 \times 10^{-6} = \mathbf{3.3}$ N.
- (c) Alone on k_1: $X = F_0/(k_1 - \omega^2 m_1) = -21.8$ µm, and the floor would feel $k_1 X = 4.4$ N; the frame adds a little isolation.
- With mode 2 at 25 Hz, the running speed would excite it: the frame amplitude and floor force would rise sharply, so every mode must be checked, not only the first.

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