

## Digital Signal Processing

Curated undergraduate study track (not accredited, no external exam). Original explanations and practice only, following the reviewed course handout.

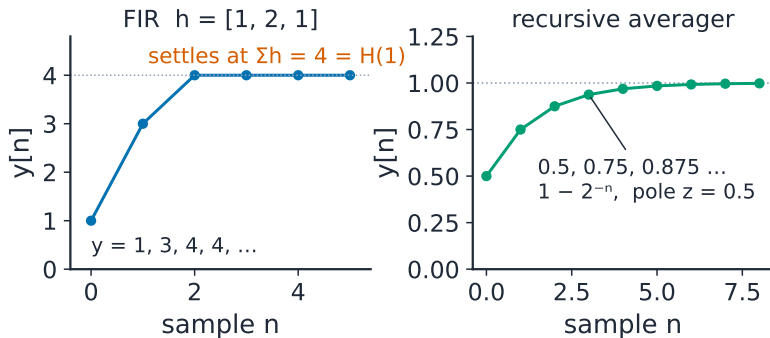
## Refounding signal theory in samples

- Foundations first (systems, **DTFT/DFT**, **FFT**), then filters (FIR windows, IIR bilinear), then rates and a verified project.
- Every design measured against its specification; every speedup an operation count, not folklore.

## Discrete systems: one list decides everything

- 1 **FIR**  $h = [1, 2, 1]$  on a unit step:  $y = 1, 3, 4, 4 \dots$  settling at  $\sum h = 4$ , the DC gain  $H(1)$ ; a pulse  $[1,1,1]$  reads  $1, 3, 4, 3, 1$  - the taps mirrored.
- 2 **IIR** The averager  $y[n] = (y[n-1] + x[n])/2$  steps up  $0.5, 0.75, 0.875, 0.9375 \dots = 1 - 2^{-n}$ , because its pole sits at  $z = 0.5$  - it approaches, never arrives.
- 3 **Stability** BIBO asks one question of the taps: does  $\sum |h[n]|$  converge? For rational systems that is exactly all poles inside  $|z| = 1$ .

# The sliding table and the pole

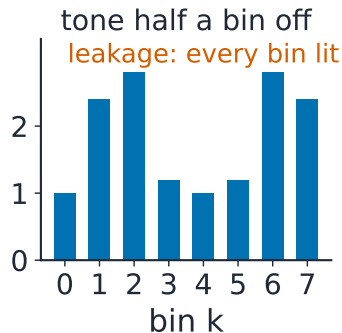
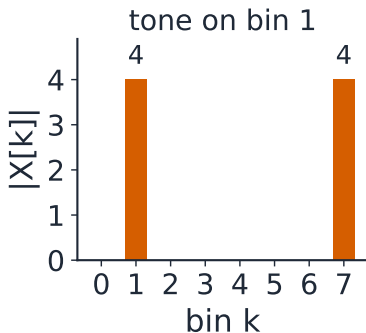


The FIR settles in three samples at  $\Sigma h = 4$ ; the recursive averager closes on 1 as  $1 - 2^{-n}$  from its  $z = 0.5$  pole.

## DTFT and DFT: sampling in frequency

- 1 **On-bin** 8-point **DFT** of a cosine on bin 1:  $|X| = 4$  at bins 1 and 7, zero elsewhere  
- the periodic copies join seamlessly.
- 2 **Off-bin** Half a bin off: 2.40, 2.80, 1.19, 1.0 ... in every bin - the rectangle sidelobes at  $-13$  dB paint the spectrum; Hamming drops them to  $-43$  dB.
- 3 **Padding** Zero-padding interpolates the same spectrum - it resolves nothing.  
**Resolution** was decided when you chose how long to measure.

## Two spikes become every bin

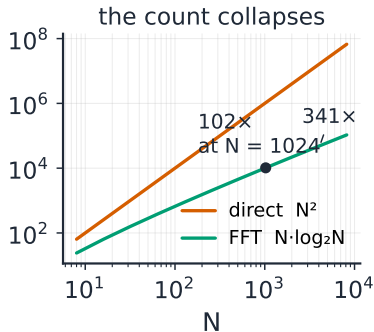
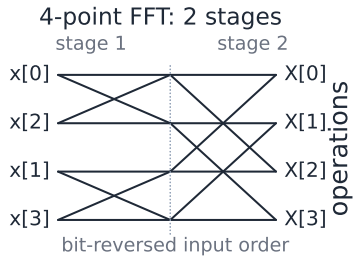


On bin 1: two clean spikes of 4. Half a bin off: every bin lit - **leakage** is the rectangular window speaking.

## The FFT: same answer, fewer operations

- 1 **Count**  $N = 1024$ : direct  $N^2 = 1\,048\,576$  operations, **FFT**  $N \cdot \log_2 N = 10\,240$  -  $102\times$  fewer;  $N = 4096$ :  $341\times$ , and the gap widens with  $N$ .
- 2 **Real time** 1 Hz bins at  $f_s = 1$  kHz:  $N = 1024$  gives 0.977 Hz; at 10 ns per operation the transform takes  $\sim 100$   $\mu$ s against a 1.024 s record.
- 3 **Map** Bin 100 of that **DFT** is  $100 \times 0.977 = 97.66$  Hz - the bin-to-hertz map  $k \cdot f_s/N$ , the classic lab error when forgotten.

# The butterfly ladder

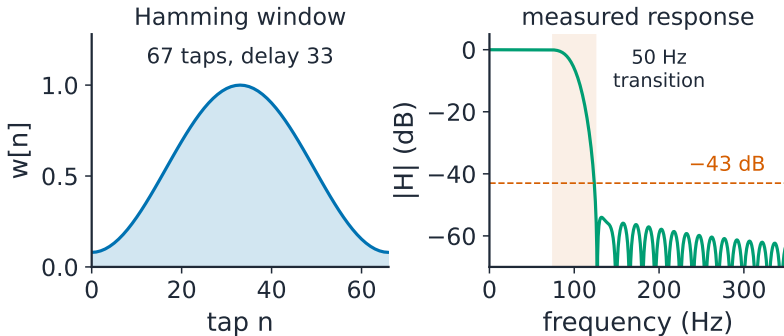


Two stages of butterflies build a 4-point **FFT**; at  $N = 1024$  the count collapses 102× and keeps falling.

## FIR: the window is the fate

- 1 **Size** Hamming at  $f_s = 1$  kHz, 50 Hz transition:  $N = 3.3 \cdot 1000/50 = 66$  - take 67 (odd, Type-I), delay 33 samples;  $N = 31$  gives 106 Hz: the designs bracket the spec.
- 2 **Verify** Measure the response from the taps: stopband at or below the  $-43$  dB peak sidelobe, passband ripple inside  $\pm 0.05$  dB - a design that measures  $-37$  dB has the wrong window or the wrong  $N$ .
- 3 **Optimal** Windows are convenient, not optimal - the alternation theorem behind Parks-McClellan makes equiripple the optimum.

## Window in, response out

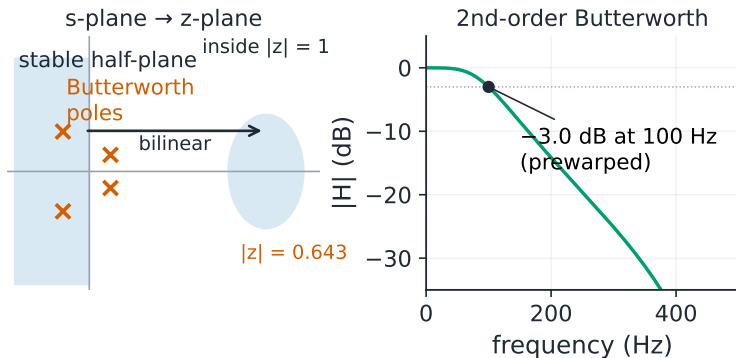


67 Hamming taps: 50 Hz transition, stopband floor at the promised  $-43$  dB, delay 33 samples.

## IIR: analog lore, one substitution

- 1 **Biquad** 2nd-order Butterworth at 100 Hz,  $f_s = 1$  kHz: **prewarp**  $\omega_a = 2000 \cdot \tan(0.1\pi) = 649.8$  rad/s,  $K = 3.078$ ,  $H(z) = (1+2z^{-1}+z^{-2})/(14.82 - 16.94z^{-1} + 6.12z^{-2})$ .
- 2 **Verify** DC gain  $4/4 = 1.000$ ;  $|H|$  at 100 Hz =  $-3.0$  dB - the cutoff lands exactly; poles at  $|z| = 0.643$ , inside by the mapping itself.
- 3 **Trap** Skip the prewarp ( $K = 1$ ) and the same prototype cuts at 250 Hz, reading a complacent  $-0.05$  dB at 100 Hz - silently off-spec.

# The left half-plane moves in

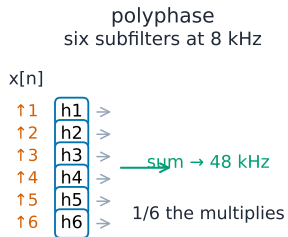
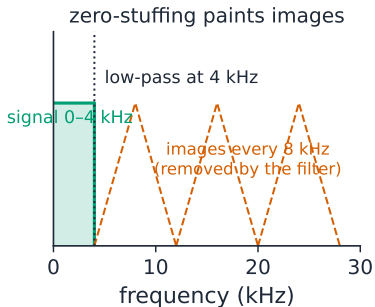


The stable half-plane folds inside  $|z| = 1$  (poles land at 0.643) and the **prewarp** puts the  $-3$  dB point on 100 Hz exactly.

## Multirate: work where it is cheap

- 1 **Up** 8 kHz speech to 48 kHz: interpolate by 6, low-pass at 4 kHz kills images at 8k, 16k, 40k; **polyphase** runs 6 subfilters at 8 kHz - 1/6 the multiplies.
- 2 **Ratio** 48  $\rightarrow$  72 kHz is  $L = 3$ ,  $M = 2$ : upsample to 144 kHz, ONE low-pass at 24 kHz stops the  $\uparrow 3$  images and the  $\downarrow 2$  aliases, keep every second sample.
- 3 **Gain** Zero-stuffing loses a factor  $L$  of amplitude - compensate it in the filter; forgetting the gain of  $L$  is the classic multirate bug.

# Images every 8 kHz



Stuffing 5 zeros paints images every 8 kHz; one low-pass at 4 kHz erases them;  
**polyphase** pays 1/6 the multiplies.

## The project: specify, verify, measure

- 1 **Spec** 1 Hz bins at  $f_s = 1$  kHz needs  $N \geq 1000$  - already 1.0 s against the  $<1$  s latency line: the conflict surfaces at design time, exactly when you want it.
- 2 **Stages** Filter  $\rightarrow$  window  $\rightarrow$  **FFT**  $\rightarrow$  threshold, each verified alone: a defect found in a stage costs an hour; in the integrated system, a week of bisection.
- 3 **Report** The final table lists spec, measured, margin per line - 5 of 6 and honest beats 6 of 6 claimed with 4 measured.

## Four blocks and a margin table

verify each stage alone — then measure the whole



| spec   | measured    | margin |
|--------|-------------|--------|
| 1 Hz   | 0.977 Hz    | ok     |
| <1 s   | 1.024 s     | ✗      |
| -10 dB | -10 + 27 dB | ok     |

Each block carries its own check; the pipeline ends in the spec-versus-measured margin table with no silent goals.

## Check yourself

- **FIR**:  $h = [1, 2, 1]$  on a unit step settles at? (4 - the tap sum, the DC gain.)
- **DFT**: 1 Hz bins at  $f_s = 1$  kHz need  $N = ?$  (1000 - one second of data.)
- **FFT**:  $N = 1024$  speedup over direct? ( $102\times$  -  $N^2$  against  $N \cdot \log_2 N$ .)
- **IIR**: skip the prewarp for 100 Hz at 1 kHz and it cuts at? (250 Hz - the classic silent off-spec.)