

Exercise walk-through

12.3

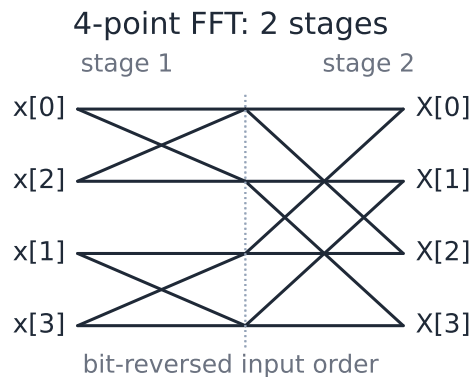
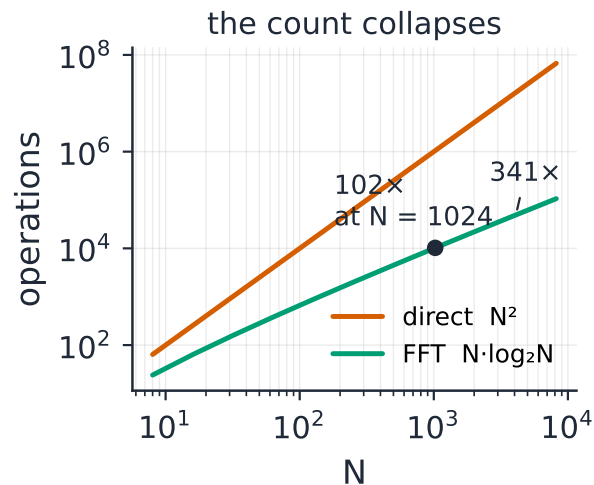
The FFT

University Electrical Engineering

The FFT: same answer, fewer operations

- 1 **Count** $N = 1024$: direct $N^2 = 1\,048\,576$ operations, **FFT** $N \cdot \log_2 N = 10\,240$ - $102\times$ fewer; $N = 4096$: $341\times$, and the gap widens with N .
- 2 **Real time** 1 Hz bins at $f_s = 1$ kHz: $N = 1024$ gives 0.977 Hz; at 10 ns per operation the transform takes ~ 100 μ s against a 1.024 s record.
- 3 **Map** Bin 100 of that **DFT** is $100 \times 0.977 = 97.66$ Hz - the bin-to-hertz map $k \cdot f_s/N$, the classic lab error when forgotten.

The butterfly ladder



Two stages of butterflies build a 4-point **FFT**; at $N = 1024$ the count collapses 102 $\times$ and keeps falling.

Four blocks and a margin table

verify each stage alone — then measure the whole



spec	measured	margin
1 Hz	0.977 Hz	ok
<1 s	1.024 s	✗
-10 dB	-10 + 27 dB	ok

Each block carries its own check; the pipeline ends in the spec-versus-measured margin table with no silent goals.

Exercises 2.1-2.3

2.1 How many stages and butterflies for an $N = 64$ radix-2 FFT? [3]

2.2 At $f_s = 8$ kHz with $N = 512$: bin width, and the frequency of bin 90. [3]

2.3 Direct DFT of $N = 256$: operation count, and the FFT count. [3]

Exercises 2.1-2.3

2.1 How many stages and butterflies for an $N = 64$ radix-2 FFT? [3]

2.1

- $\log_2 64 = 6$ stages, each $N/2 = 32$ butterflies.

2.2 At $f_s = 8$ kHz with $N = 512$: bin width, and the frequency of bin 90. [3]

2.2

- $\Delta f = 8000/512 = 15.625$ Hz.
- Bin 90: $90 \times 15.625 = 1406.25$ Hz.

2.3 Direct DFT of $N = 256$: operation count, and the FFT count. [3]

2.3

- Direct $256^2 = 65\,536$; FFT $256 \times 8 = 2048 - 32\times$ fewer.

Exercise 3.1

3.1 A real-time analyser samples at $f_s = 48$ kHz and must update every 21.3 ms (1024-sample hop).

Compute the bin width. Compute the FFT operation count and, at 5 ns per operation, the compute time per transform; state the margin against the hop. Compute which bin a 1 kHz tone lands in (and its exact frequency). [8]

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3.1

- $\Delta f = 48\,000/1024 = 46.875$ Hz.
- FFT count $1024 \times 10 = 10\,240$ operations.
- At 5 ns: $51.2\ \mu s$ per transform.
- Margin: $51.2\ \mu s$ against 21.3 ms – about $400\times$ headroom ✓.
- 1 kHz: $1000/46.875 = 21.33 \rightarrow$ bin 21, exact frequency $21 \times 46.875 = 984.375$ Hz (0.984 kHz).

Exercise 3.2

3.2 A colleague proposes replacing an $N = 1024$ FFT by two $N = 512$ FFTs plus 512 combining butterflies to halve the latency.

Check the operation count of the proposal against one $N = 1024$ FFT. Explain where the saving (if any) comes from, and what Danielson–Lanczos says the structure actually is. State one reason the split helps pipeline implementations. [7]

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3.2

- Two 512-FFTs: $2 \times 512 \times 9 = 9216$, plus 512 butterflies = 9728 operations – versus 10 240: a 5% saving only.
- Danielson–Lanczos: one 1024 DFT IS two 512 DFTs plus N twiddle combines – the proposal reinvents the first FFT stage.
- Splitting helps pipelining: shorter transforms mean smaller, shallower butterfly arrays with cleaner memory locality.