

Exercise walk-through

12.2

DTFT and DFT

University Electrical Engineering

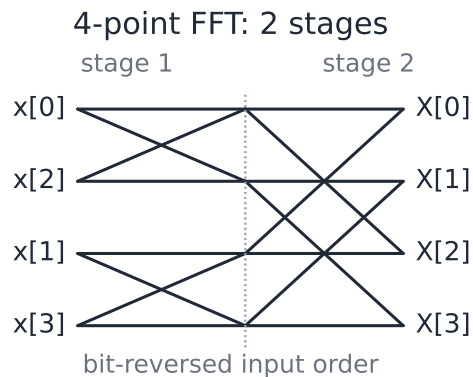
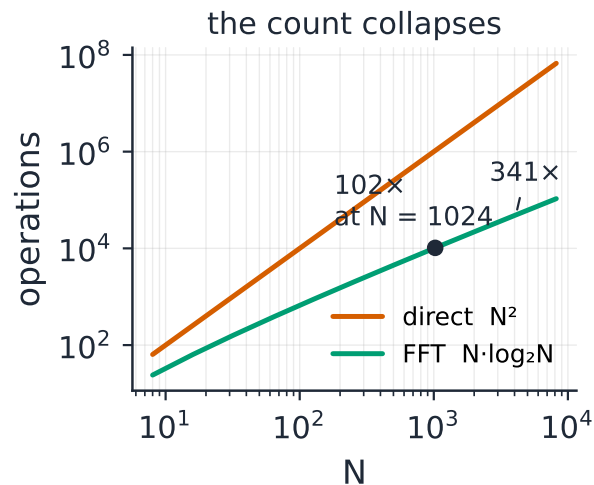
DTFT and DFT: sampling in frequency

- 1 **On-bin** 8-point **DFT** of a cosine on bin 1: $|X| = 4$ at bins 1 and 7, zero elsewhere - the periodic copies join seamlessly.
- 2 **Off-bin** Half a bin off: 2.40, 2.80, 1.19, 1.0 ... in every bin - the rectangle sidelobes at -13 dB paint the spectrum; Hamming drops them to -43 dB.
- 3 **Padding** Zero-padding interpolates the same spectrum - it resolves nothing. **Resolution** was decided when you chose how long to measure.

The FFT: same answer, fewer operations

- 1 **Count** $N = 1024$: direct $N^2 = 1\,048\,576$ operations, **FFT** $N \cdot \log_2 N = 10\,240$ - $102\times$ fewer; $N = 4096$: $341\times$, and the gap widens with N .
- 2 **Real time** 1 Hz bins at $f_s = 1$ kHz: $N = 1024$ gives 0.977 Hz; at 10 ns per operation the transform takes ~ 100 μ s against a 1.024 s record.
- 3 **Map** Bin 100 of that **DFT** is $100 \times 0.977 = 97.66$ Hz - the bin-to-hertz map $k \cdot f_s/N$, the classic lab error when forgotten.

The butterfly ladder



Two stages of butterflies build a 4-point **FFT**; at $N = 1024$ the count collapses 102 $\times$ and keeps falling.

Exercises 2.1-2.2

2.1 At $f_s = 2$ kHz, what N gives 4 Hz bins? [3]

2.2 A tone sits at bin 3 of an on-bin 8-point DFT with amplitude 2. Give $|X[k]|$ for all bins. [3]

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2.1

- $N = f_s / \Delta f = 2000 / 4 = 500$.

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2.2

- Amplitude 2 $\rightarrow$ each exponential carries amplitude 1, summing over N samples:
 $|X[3]| = |X[5]| = A N / 2 = 2 \times 4 = 8$, all other bins 0.

Exercise 2.3

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- $\Delta f \leq 3$ Hz needs $N \geq 1000/3 = 334 \rightarrow N = 334$ or more (0.334 s).
- Padding only interpolates the same spectrum – no new information to separate the tones.

Exercise 3.1

3.1 A spectrum analyser at $f_s = 1$ kHz must resolve the 50 Hz mains tone from a 55 Hz interferer.

State the minimum N and the record length in seconds. Compute the bin width for $N = 256$ and state whether the two tones are resolvable. Explain what zero-padding to 2048 changes and what it cannot change. [7]

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3.1

- $\Delta f < 5$ Hz separation needs $N > 1000/5 = 200$: minimum $N = 200$ (0.2 s).
- $N = 256$: $\Delta f = 3.906$ Hz.
- $3.906 < 5 \rightarrow$ resolvable $\checkmark$ (bins $50/3.906 = 12.8$ and $55/3.906 = 14.08$ – different bins).
- Padding to 2048 draws the same underlying spectrum with $8\times$ more points (prettier plot); it cannot create the frequency separation that the 0.2 s record already fixed.

Exercise 3.2

3.2 An 8-point DFT of $x[n] = \cos(2\pi \times 1.5 \times n/8)$ is computed with a rectangular window.

Explain why every bin is non-zero. Give the two largest bins and their magnitudes. The same tone is now windowed with Hamming: state what happens to the sidelobe level and to the main-lobe width, and the consequence for resolving two nearby tones. [8]

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3.2

- Sampling in frequency $\Rightarrow$ periodic time: a 1.5-bin tone's periodic copies do not align, so the mismatch smears into every bin – leakage.
- Largest: $|X[2]| = |X[6]| = 2.80$, then $|X[1]| = |X[7]| = 2.40$.
- Hamming: sidelobes drop from -13 dB to -43 dB (small tones beside large ones reappear).
- Main lobe widens (about $2\times$ the rectangle): two nearby tones now blur together – the trade is dynamic range against resolution.