

Exercise walk-through

12.1

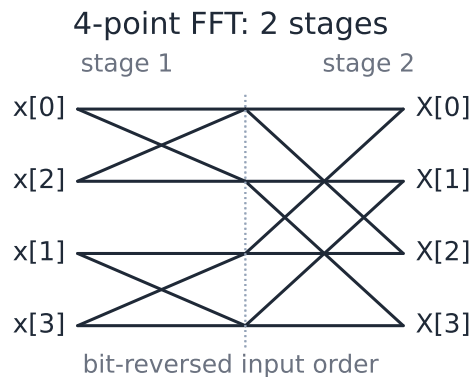
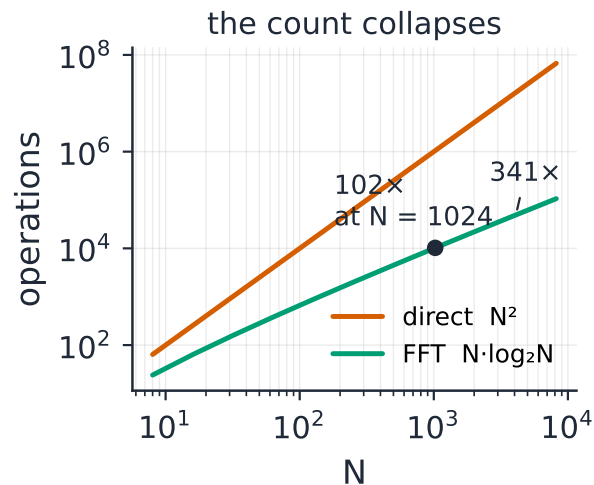
Discrete-time signals and systems review

University Electrical Engineering

The FFT: same answer, fewer operations

- 1 **Count** $N = 1024$: direct $N^2 = 1\,048\,576$ operations, **FFT** $N \cdot \log_2 N = 10\,240$ - $102\times$ fewer; $N = 4096$: $341\times$, and the gap widens with N .
- 2 **Real time** 1 Hz bins at $f_s = 1$ kHz: $N = 1024$ gives 0.977 Hz; at 10 ns per operation the transform takes ~ 100 μ s against a 1.024 s record.
- 3 **Map** Bin 100 of that **DFT** is $100 \times 0.977 = 97.66$ Hz - the bin-to-hertz map $k \cdot f_s/N$, the classic lab error when forgotten.

The butterfly ladder



Two stages of butterflies build a 4-point **FFT**; at $N = 1024$ the count collapses 102 $\times$ and keeps falling.

The project: specify, verify, measure

- 1 **Spec** 1 Hz bins at $f_s = 1$ kHz needs $N \geq 1000$ - already 1.0 s against the <1 s latency line: the conflict surfaces at design time, exactly when you want it.
- 2 **Stages** Filter $\rightarrow$ window $\rightarrow$ **FFT** $\rightarrow$ threshold, each verified alone: a defect found in a stage costs an hour; in the integrated system, a week of bisection.
- 3 **Report** The final table lists spec, measured, margin per line - 5 of 6 and honest beats 6 of 6 claimed with 4 measured.

Exercises 2.1·2.3

2.1 Compute $y[n]$ for $h = [1, 3, 2]$ and $x = [1, 1, 1]$. [3]

2.2 A filter has $h[n] = 0.4^n$ for $n \geq 0$. Is it BIBO stable? Justify with the sum. [3]

2.3 Give the DC gain of $h = [2, -1, 2]$ and check it against the steady step output. [3]

Exercises 2.1-2.3

2.1 Compute $y[n]$ for $h = [1, 3, 2]$ and $x = [1, 1, 1]$. [3]

2.1

- Diagonals: $y = 1, 4, 6, 5, 2$.

2.2 A filter has $h[n] = 0.4^n$ for $n \geq 0$. Is it BIBO stable? Justify with the sum. [3]

2.2

- $\sum 0.4^n = 1/(1 - 0.4) = 1.667$ converges ✓ stable.

2.3 Give the DC gain of $h = [2, -1, 2]$ and check it against the steady step output. [3]

2.3

- $H(1) = 2 - 1 + 2 = 3$; steady step output 3 ✓.

Exercise 3.1

3.1 A 3-tap FIR has $h = [1, a, 1]$.

For $a = 2$ compute the step response and its settling value. For which a does the step response stay non-negative at every sample? State the causality and BIBO status of this filter, with reasons.

[7]

Exercise 3.1

3.1 A 3-tap FIR has $h = [1, a, 1]$.

For $a = 2$ compute the step response and its settling value. For which a does the step response stay non-negative at every sample? State the causality and BIBO status of this filter, with reasons. [7]

3.1

- $a = 2$: step response $1, 3, 4, 4, \dots$, settling at $\sum h = 4$.
- Non-negative everywhere needs $y[1] = 1 + a \geq 0$ and $y[3] = 2 + a \geq 0$: the binding sample is $n = 1$, so $a \geq -1$.
- Causal: $h[n] = 0$ for $n < 0$ ✓ (finite taps).
- BIBO: finite $\sum |h| = |1| + |a| + |1|$ always converges ✓ every FIR is unconditionally stable.

Exercise 3.2

3.2 The leaky integrator $y[n] = \alpha y[n-1] + x[n]$ with $y[-1] = 0$ and $0 < \alpha < 1$.

Compute the impulse response, the step response at $n = 0, 1, 2, 3$ for $\alpha = 0.5$, and its closed form. State the pole, the stability condition on α , and how many samples until the step response is within 1% of its limit when $\alpha = 0.5$. [8]

Exercise 3.2

3.2 The leaky integrator $y[n] = \alpha y[n-1] + x[n]$ with $y[-1] = 0$ and $0 < \alpha < 1$.

Compute the impulse response, the step response at $n = 0, 1, 2, 3$ for $\alpha = 0.5$, and its closed form. State the pole, the stability condition on α , and how many samples until the step response is within 1% of its limit when $\alpha = 0.5$. [8]

3.2

- Impulse response $h[n] = \alpha^n, n \geq 0$.
- Step: 1, 1.5, 1.75, 1.875 ✓.
- Closed form $y[n] = (1 - \alpha^{n+1})/(1 - \alpha)$; at $\alpha = 0.5$: $2 - 2^{-n}$ ✓ (limit 2).
- Pole at $z = \alpha$; stable for $0 < \alpha < 1$ ($\sum \alpha^n$ converges).
- Within 1% of limit 2: remaining gap $2^{-n} \leq 0.02 \Rightarrow n \geq 6$ samples.