Electromagnetics

University Electrical Engineering

What this course is, and how to use it

This is the seventh course in University Electrical Engineering: **electromagnetics** – the physics under every circuit, antenna and motor. Circuits hide the fields inside wires; this course opens the box.

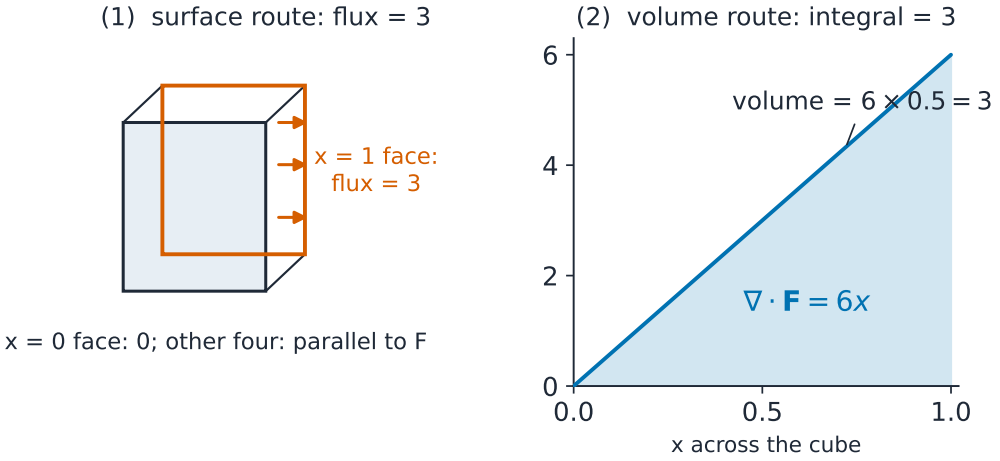
- Unit 7.1 reviews the vector calculus toolkit (EE02's operators, now with coordinates and symmetries).
- Units 7.2–7.4 build statics: electrostatics, magnetostatics, and the assembled Maxwell equations.
- Units 7.5–7.6 propagate waves and feed them down transmission lines.
- Unit 7.7 opens waveguides and antennas.

The running story is one link: a 3 GHz radio connection between two boards on a bench – its fields, its wave, its cable, its guide, its antennas. Every field answer below is computed two ways (Gauss vs direct integration; Ampère vs Biot-Savart), and every wave answer is checked against a limiting case (free space, perfect conductor, DC).

Vector field tools

Why it matters. A phone's GPS receiver hears −130 dBm; whether that works is decided by divergence and curl. The operators of EE02 are the grammar; here they take coordinates and learn the symmetry shortcut that makes real geometry tractable.

The idea, with a picture. A field has an outflow per volume (the **divergence**) and a circulation per area (the **curl**). Before touching an integral, test the symmetry of the source: full spherical symmetry → Gauss's law on a sphere ($dA = r^2 \sin\theta \, d\theta \, d\varphi$); cylindrical → a coaxial cylinder ($dA = r \, d\varphi \, dl$); none → direct integration along the field formula.



One cube, two routes: integrate 6x over the box, or count the flux through its faces

The derivation. The divergence theorem

$$\oint_S \mathbf{F} \cdot d\mathbf{A} = \int_V (\nabla \cdot \mathbf{F}) \, dV$$

says the field's net outflow through a closed surface equals the volume integral of its sources. For a symmetric source, the surface integral collapses to $E \times A$; solving for the field is then one line – that one line is every Gauss-law example in this course.

> **Worked example 1 – divergence with coordinates.** Take $\mathbf{F} = (3x^2, 0, 0)$: $\nabla \cdot \mathbf{F} = \partial(3x^2)/\partial x = 6x$, so at the point $(2, 0, 0)$ the divergence is 12 per unit volume. Check at the point $(-2, 0, 0)$: the field still points along $+x$ ($3x^2$ is never negative), yet the divergence is -12 – the sign tracks how the field changes (the source density), not which way it points.

> **Worked example 2 – the theorem on a cube.** The flux of that $\mathbf{F}$ out of the unit cube $[0, 1]^3$, by the volume route: $\int_0^1 \int_0^1 \int_0^1 6x \, dx \, dy \, dz = 6 \cdot \frac{1}{2} = 3$. By the surface route: only the $x = 1$ face carries flux ($3 \cdot 1 \cdot 1 = 3$; the $x = 0$ face has $3x^2 = 0$, the four others are parallel to $\mathbf{F}$) – total 3 ✓. Both routes, one number: the theorem is an accounting identity you can always afford to check.

Common mistakes. Integrating r^2 but differentiating $1/r^2$ (most Gauss examples hide one such step – write both explicitly). Forgetting the sphere's $dA = r^2 \sin\theta \, d\theta \, d\varphi$ has an r^2 that cancels the field's $1/r^2$ – that cancellation IS the method. And applying Gauss's law to asymmetric sources: no symmetry, no $E \times A$, no shortcut.

Summary. Divergence = outflow density, curl = circulation density; symmetry chooses the surface; the divergence theorem is the two-route check that makes every answer here self-verifying.

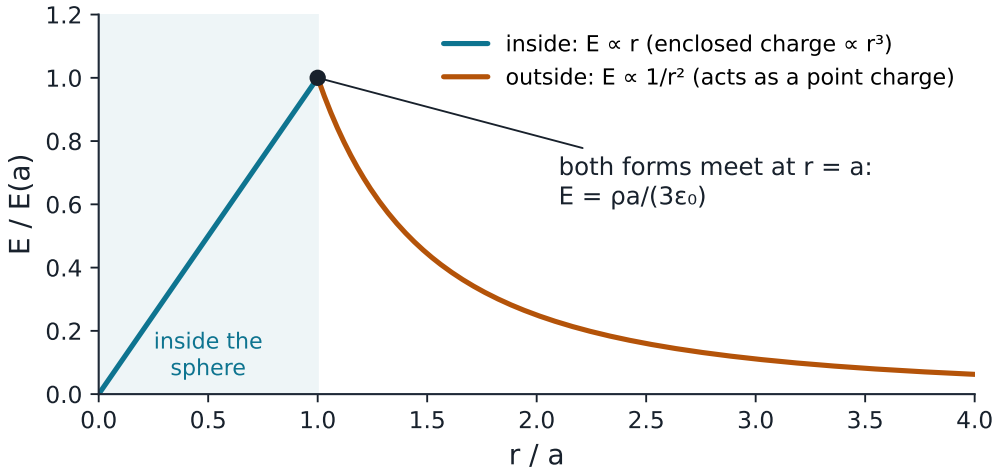
Electrostatics

Why it matters. Every capacitor, every insulation failure, every touch sensor is an electrostatics problem wearing a component label.

The idea, with a picture. The Coulomb field of a point charge, $E = q/(4\pi\varepsilon_0 r^2)\,\hat{r}$, gathers into **Gauss's law** :

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0},$$

and symmetry converts the integral into arithmetic: for a spherically symmetric source, $E \cdot 4\pi r^2 = Q_{enc}/\varepsilon_0$.



Uniform sphere: $E \propto r$ inside, $E \propto 1/r^2$ outside, continuous at the surface

The derivation. Inside a uniformly charged sphere of radius a and density ρ, the enclosed charge grows as r^3 while the Gaussian sphere's area grows as r^2:

$$E(r)\cdot 4\pi r^2 = \frac{\rho\cdot\frac{4}{3}\pi r^3}{\varepsilon_0} \;\Longrightarrow\; E = \frac{\rho r}{3\varepsilon_0}\ (r<a), \qquad E = \frac{\rho a^3}{3\varepsilon_0 r^2}\ (r>a).$$

The **potential** stacks the field: $V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l}$; the **capacitance** of an isolated sphere follows, $C = Q/V = 4\pi\varepsilon_0 a$ – about 1.1 pF per cm of radius, why free-space objects barely couple. At a dielectric interface the **boundary conditions** are: tangential E continuous, normal $D = \varepsilon E$ jumping by any free surface charge.

> **Worked example 1 – the sphere, both regions.** With $\rho = 1\ \mu\text{C/m}^3$ and $a = 2$ cm: at the surface $r = a$, both formulas give $E = \rho a/3\varepsilon_0 = (10^{-6}\cdot 0.02)/(3\times 8.854\times 10^{-12}) = 753$ V/m ✓ one value, two routes. At $r = 4$ cm (double the radius) the outside form gives $753/4 = 188$ V/m – the $1/r^2$ falloff, one octave out.

> **Worked example 2 – escalate to the dielectric.** Keep the same free charge density and let a homogeneous, linear dielectric fill all space, inside and outside the charged region. With $\varepsilon_r = 4$, the field everywhere drops by 4× and the surface value becomes 188 V/m. The **polarisation** changes E, while $D = \varepsilon E$ is unchanged for the same free charge. The bound charge inside the sphere is $Q_{bound} = -(1-1/\varepsilon_r)Q_{free} = -0.75Q_{free}$. It is distributed through the charged volume. There is no extra bound surface charge at $r = a$, because the dielectric is the same on both sides and the polarisation is continuous there. A dielectric sphere surrounded by vacuum is a different boundary-value problem.

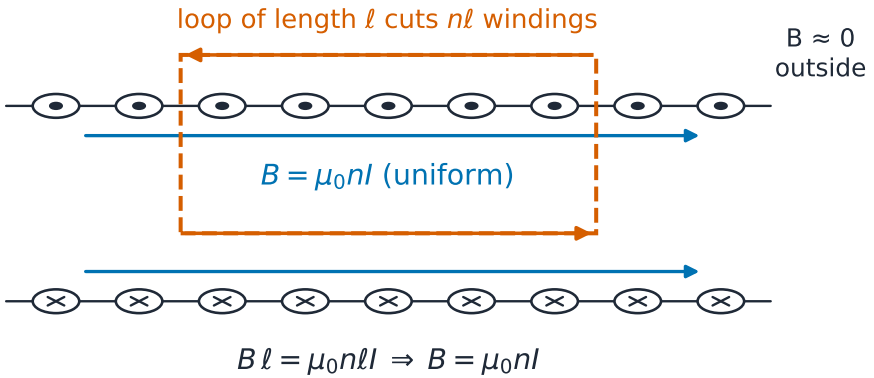
Common mistakes. Using $4\pi r^2$ where the geometry is a cylinder ($2\pi rl$). Mixing ε_0 and $\varepsilon = \varepsilon_0\varepsilon_r$ mid-problem – pick one convention per problem. And applying the outside (point-charge) form inside the sphere: the enclosed charge, not the total charge, is what Gauss sees.

Summary. Gauss's law plus symmetry: $E \times A = Q_{enc}/\varepsilon$; the sphere grows linearly inside, falls as $1/r^2$ outside, agrees at the surface. A dielectric divides fields by ε_r and stores the difference as bound charge.

Magnetostatics

Why it matters. Motors, relays, MRI magnets and every inductor begin with the same law pair: Biot-Savart for the general case, Ampère when symmetry is kind.

The idea, with a picture. Currents make fields. **Biot-Savart law** integrates every current element's contribution; **Ampère's law** $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{enc}$ wraps the field around the current. The symmetry test from Unit 7.1 decides which to use.



The rectangular Amperian loop cuts the windings; the outside long side contributes zero

The derivation. For an infinite solenoid with n turns per metre, take a rectangular Amperian loop with one long side inside (length l), one outside, and the short sides crossing the windings. The enclosed current is nlI; the outside field of an infinite solenoid vanishes by cancellation, and the short sides contribute nothing (field perpendicular to them). So $B \cdot l = \mu_0 nlI$:

$$B = \mu_0 nI \quad \text{(inside, uniform)}, \qquad B \approx 0 \ \text{(outside)}.$$

Magnetic flux $\Phi = \int \mathbf{B} \cdot d\mathbf{A}$ leads to **inductance** $L = N\Phi/I$; for the solenoid, $L/l = \mu_0 n^2 A$. The **Lorentz force** $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ bends moving charge into a circle of radius $r = mv/qB$ – the cyclotron orbit.

> **Worked example 1 – the solenoid priced.** With $n = 1000$ turns/m and $I = 2$ A: $B = 4\pi \times 10^{-7} \times 1000 \times 2 = 2.51$ mT inside, uniform along the axis. A Biot–Savart check would integrate a thousand loops – this is why the symmetry test comes first: one line instead of a page.

> **Worked example 2 – the orbit and the force.** A 1 μC charge at 10^6 m/s moving perpendicular to a 1 T field feels $F = qvB = 10^{-6} \times 10^6 \times 1 = 1$ N – and circles at radius $r = mv/qB$: for an electron ($m = 9.11 \times 10^{-31}$ kg), $r = (9.11 \times 10^{-31} \times 10^6)/(1.6 \times 10^{-19} \times 1) = 5.7$ μm – the cyclotron radius that sets magnetron and spectrometer geometry. Check the speed: an electron at 10^6 m/s carries 2.85 eV – comfortably non-relativistic, so the formula is honest.

Common mistakes. Forgetting Ampère's law needs the TOTAL enclosed current (loop with N turns $\to NI$). Assuming $\mathbf{B}$ is continuous across a current sheet (its tangential component jumps by $\mu_0 K$). And using the solenoid formula for a short coil: "infinite" means length $\gg$ diameter, or the field is weaker and non-uniform.

Summary. Ampère plus symmetry: $B \cdot l = \mu_0 I_{enc}$; the infinite solenoid concentrates flux, $B = \mu_0 nI$, $L/l = \mu_0 n^2 A$; moving charge circles at $r = mv/qB$.

Maxwell's equations

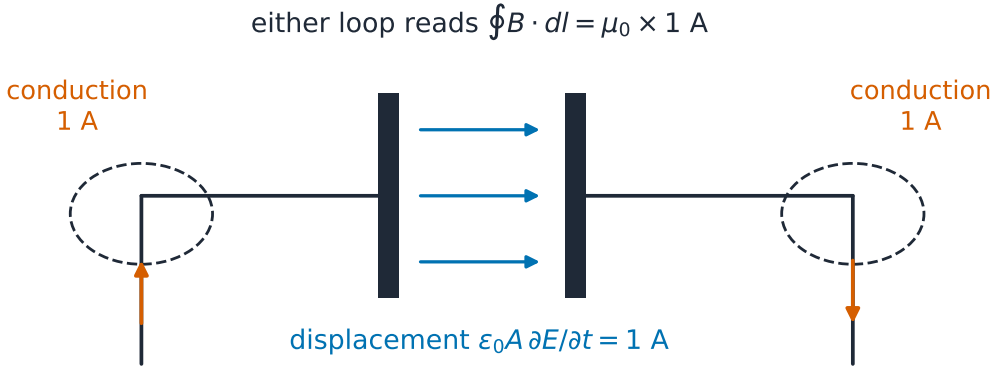
Why it matters. Every radio, every photon, every charger's wireless coupling rides on the one term Maxwell added to close the loop.

The idea, with a picture. Statics + **displacement current** completes the set:

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \, \partial \mathbf{E}/\partial t$$

The four boxes are a loop: circulating $\mathbf{E}$ makes changing $\mathbf{B}$, circulating $\mathbf{B}$ makes changing $\mathbf{E}$ – self-sustaining waves.



The loop integral cannot depend on which surface you span it with – displacement current makes it true

The derivation. Charge conservation $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$ forces the correction: take the divergence of the original Ampère law ($\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$) and the left side is identically zero while the right side is $-\mu_0 \partial\rho/\partial t$ – nonzero whenever charge piles up. Adding $\mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t$ repairs it, because Gauss's law turns that term's divergence into exactly $+\mu_0 \partial\rho/\partial t$. The loop closes.

> **Worked example 1 – the capacitor's missing ampere.** A parallel-plate capacitor charges at $dQ/dt = 1$ A. Between the plates no conduction current exists, yet an Ampèrian loop around the wire must read the same 1 A whichever surface you span it with. Between the plates $E = Q/(\varepsilon_0 A)$, so $\varepsilon_0 A \cdot \partial E/\partial t = dQ/dt = 1$ A ✓ – the displacement current IS the conduction current, numerically.

Worked example 2 – escalate to the wave speed. In source-free space ($\rho = 0$, $\mathbf{J} = 0$), take the curl of the **E** curl equation and substitute the **B** one:

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0\,\partial^2\mathbf{E}/\partial t^2,$$

the wave equation with speed $c = 1/\sqrt{\mu_0\varepsilon_0}$. Insert the numbers: $1/\sqrt{\mu_0\varepsilon_0} = 3.0 \times 10^8$ m/s – light, from two constants measured on a bench. The link budget of the running story begins here.

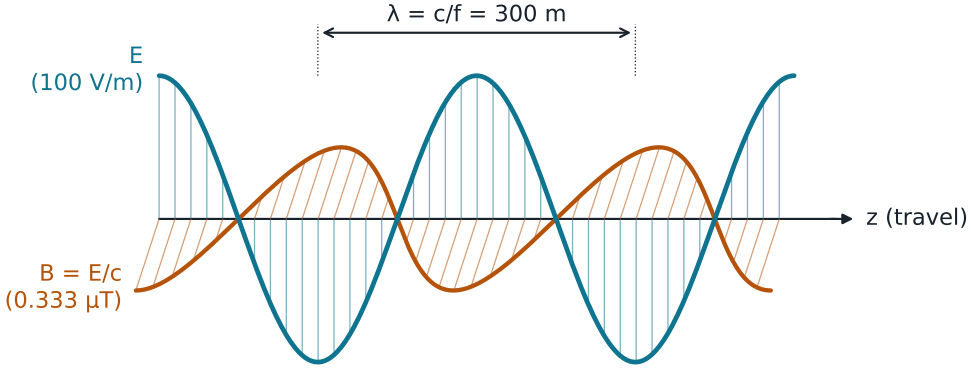
Common mistakes. Treating displacement current as optional bookkeeping (it is what makes the equations consistent and what carries the wave). Forgetting which equation has the minus sign (Faraday's: it is Lenz's law, the induced field opposing the change). And applying the static forms to antennas: radiation IS the correction term doing its work.

Summary. Four equations, one loop; displacement current closes charge conservation and, in vacuum, yields the wave equation at $c = 1/\sqrt{\mu_0\varepsilon_0}$.

Plane waves

Why it matters. Wi-Fi, radar, sunlight and the bench link's 3 GHz carrier are all the same solution: the plane wave.

The idea, with a picture. In source-free space the wave equation's simplest solution is a **plane wave** : **E**, **B** and the propagation direction mutually perpendicular, in phase, with $B = E/c$. The **intrinsic impedance** $\eta_0 = \sqrt{\mu_0/\varepsilon_0} = 377\ \Omega$ of free space sets the ratio of E to H; the **Poynting vector** $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ points along the propagation with $S = E^2/\eta_0$ (instantaneous; with rms values of E this is the average power density, while a sinusoid of amplitude E_0 averages $E_0^2/2\eta_0$).



E and B in phase and at right angles; λ = 300 m at 1 MHz

The derivation. For $\mathbf{E} = E_0\cos(\omega t - kz)\,\hat{x}$, Faraday's law gives $\partial B/\partial t = -\partial E/\partial z$, integrating to $B = (k/\omega)E = E/c$. The power density follows from the impedance: $H = E/\eta_0$, so $S = E \times H = E^2/\eta_0$ – one number per wave.

Worked example 1 – reading the numbers off. A 100 V/m (rms) wave at 1 MHz: $B = 100/(3 \times 10^8) = 0.333\ \mu$T; wavelength $\lambda = c/f = 300$ m; power density $S = 100^2/377 = 26.5$ W/m^2. Check the impedance route: $H = 100/377 = 0.265$ A/m, and $E \times H = 26.5$ ✓ – the two routes agree because they are the same ratio.

Worked example 2 – escalate to the conductor's skin. In a **good conductor** the wave decays as $e^{-\alpha z}$ with **skin depth** $\delta = \sqrt{2/(\omega\mu\sigma)}$. Copper ($\sigma = 5.8 \times 10^7$): at 50 Hz, $\delta = \sqrt{2/(2\pi \cdot 50 \cdot 4\pi \times 10^{-7} \cdot 5.8 \times 10^7)} = 9.3$ mm; at 1 GHz, $\delta = 2.1\ \mu$m – about 4500 times thinner for twenty million times the frequency ($\delta \propto 1/\sqrt{f}$ ✓). This one number explains why shields work, why RF current rides the surface, and why the bench link's 3 GHz traces need wide, smooth copper: at 3 GHz the current lives in the top 1.2 µm, so surface roughness and any plating there set the loss (gold plating protects against corrosion; it conducts worse than copper).

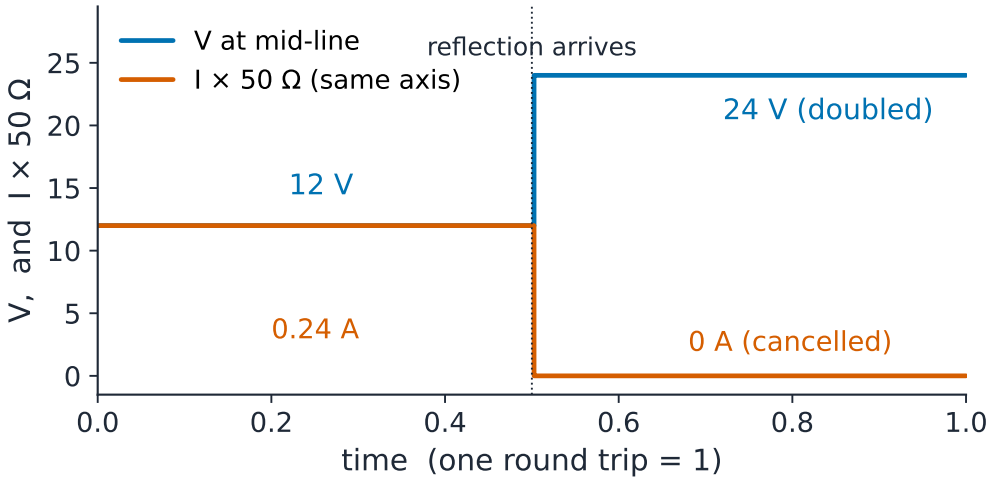
Common mistakes. Using c in a dielectric – the speed there is $1/\sqrt{\mu\varepsilon}$ (≈ 0.67c in the coax below). Mixing E/H (377 Ω in space) with E/B (c – a speed, not an impedance). And forgetting the wave's fields are in phase in a LOSSLESS medium only.

Summary. Plane wave: $E \perp B \perp \hat{k}$, $B = E/c$, $S = E^2/\eta_0$; conductors eat waves as $e^{-z/\delta}$ with $\delta = \sqrt{2/\omega\mu\sigma}$.

Transmission lines

Why it matters. The bench link's two boards are joined by 40 cm of coax; at 3 GHz that is six wavelengths – a circuit no more, a transmission line.

The idea, with a picture. The **telegrapher equations** treat the line as distributed inductance L' and capacitance C' per metre: $\partial v/\partial x = -L'\partial i/\partial t$ and $\partial i/\partial x = -C'\partial v/\partial t$. Their solutions are voltage/current waves with $v_p = 1/\sqrt{L'C'}$ and the line's characteristic impedance $Z_0 = \sqrt{L'/C'}$.



The open-end signature: voltage doubles, current goes to zero

The derivation. Differentiate the first equation in x and the second in t and eliminate $\partial^2 i/\partial x \partial t$: a wave equation for v with speed $1/\sqrt{L'C'}$. At a load Z_L, matching gives the reflection coefficient $\Gamma = (Z_L - Z_0)/(Z_L + Z_0)$: a short ($\Gamma = -1$) inverts, an open ($\Gamma = +1$) doubles the voltage and cancels the current – the **bounce diagram** tracks every bounce.

Worked example 1 – a real coax. RG-316-class coax with $L' = 250$ nH/m and $C' = 100$ pF/m: $Z_0 = \sqrt{250\times10^{-9}/100\times10^{-12}} = \sqrt{2500} = 50\ \Omega$ ✓ (the standard cable – the numbers were chosen to make it so); $v_p = 1/\sqrt{250\times10^{-9}\times100\times10^{-12}} = 2\times10^8$ m/s $= 0.67$c. The bench link's 40 cm of it is SIX wavelengths at 3 GHz ($\lambda = v_p/f = 6.7$ cm) – proving the "line, not circuit" point with the parts on the bench.

Worked example 2 – the open-ended step. Drive that 50 Ω line with a 12 V step, open-ended. The incident wave carries 12 V and $12/50 = 0.24$ A. At the open, $\Gamma = +1$: the reflected +12 V doubles the load voltage to 24 V and the reflected current −0.24 A cancels the forward current to zero – an open circuit draws no steady current ✓, and the VSWR tends to infinity, the open-circuit signature the standing-wave meter reads.

Common mistakes. Using Z_0 as a physical resistance (it is a ratio; a lossless 50 Ω line dissipates nothing). Computing λ with c instead of v_p (30% error in PTFE). And forgetting the source-side reflection: a mismatched driver re-reflects, and the bounce diagram must include it.

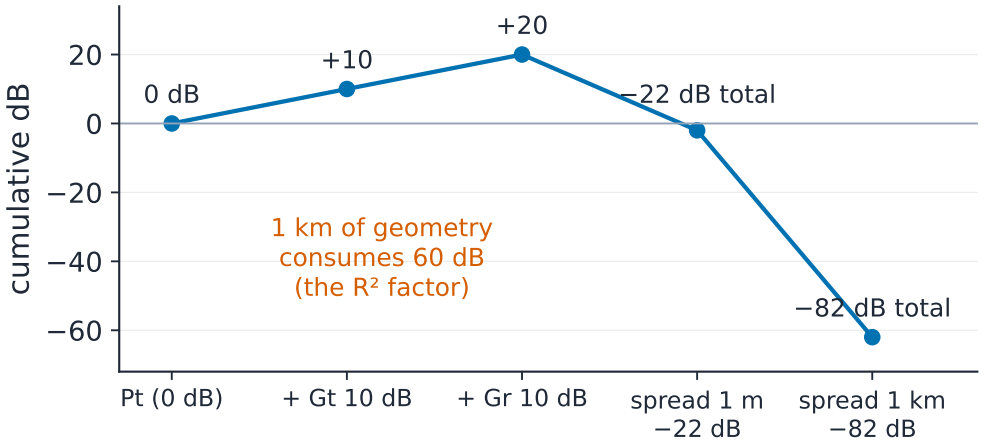
Summary. $Z_0 = \sqrt{L'/C'}$, $v_p = 1/\sqrt{L'C'}$; $\Gamma = (Z_L - Z_0)/(Z_L + Z_0)$; opens double voltage, shorts cancel it; long lines are waves, and the bounce diagram is the book-keeping.

Waveguides and antennas

Why it matters. Above ~10 GHz coax loses too much; the bench link's backhaul (and every radar) moves into hollow pipe, and the last metre of any radio link is always an antenna.

The idea, with a picture. A hollow conducting pipe guides waves only above the **cutoff** frequency: the rectangular guide's dominant mode cuts off at $f_c = c/(2a)$ (a = wide dimension). Above cutoff the guide **wavelength** stretches: $\lambda_g = \lambda/\sqrt{1-(f_c/f)^2}$ – the wave zigzags, so fewer axial cycles fit per metre of pipe. Antennas close the link: the half-wave **dipole** has gain 2.15 dBi and input resistance ≈ 73 Ω; the **Friis equation** prices the whole link:

$$P_r/P_t = G_r G_t \left(\frac{\lambda}{4\pi R}\right)^2.$$



Every dB in the chain is a design decision; the last term is pure geometry

The derivation. Between the plates of the guide, the field must fit a standing half-wave across a: $a = \lambda_c/2$ at cutoff, so $f_c = c/2a$. Along the guide, the propagation constant satisfies $k_g^2 = k^2 - (\pi/a)^2$; solving for the wavelength gives the stretch formula above. The Friis chain multiplies: transmit power × both gains × the spreading factor $(\lambda/4\pi R)^2$ – an area ratio, the antenna aperture over the sphere's area.

Worked example 1 – WR-90 at X-band. WR-90 waveguide has $a = 22.86$ mm: $f_c = 3\times10^8/(2\times0.02286) = 6.56$ GHz – X-band (8.2–12.4 GHz) rides above cutoff. At 10 GHz ($\lambda = 30$ mm): $\lambda_g = 30/\sqrt{1-0.656^2} = 39.8$ mm – 33% longer than free space. Check the limit: as $f \to f_c$, $\lambda_g \to \infty$ (the wave stops); as $f \to \infty$, $\lambda_g \to \lambda$ (the zigzag vanishes) ✓.

Worked example 2 – the bench link's budget. Two 10-dBi (G = 10) antennas, 1 m apart on the bench, at 3 GHz ($\lambda = 0.1$ m): $P_r/P_t = 10 \times 10 \times (0.1/(4\pi \times 1))^2 = 100 \times 6.33 \times 10^{-5} = 6.3 \times 10^{-3}$ – 22 dB of loss, still easy on the bench. Scale to the same boards 1 km apart: $(0.1/(4\pi \times 1000))^2 = 6.33 \times 10^{-11}$, times $100 = 6.3 \times 10^{-9}$ (-82 dB) – the link budget that decides dish sizes. One formula, two scales, and the ratio between them is pure geometry (R^2, a factor of 10^6 ✓).

Common mistakes. Using the guide's NARROW dimension for f_c (cutoff is set by the wide one). Applying λ_g as if it were λ in Friis (the free-space wavelength belongs there). And quoting dipole gain in dB without the "i": 2.15 dBi is relative to isotropic; relative to a dipole it is zero.

Summary. Guide: $f_c = c/2a$, $\lambda_g = \lambda/\sqrt{1-(f_c/f)^2}$ (stretched, infinite at cutoff). Antennas: Friis $P_r/P_t = G_r G_t(\lambda/4\pi R)^2$ – geometry decides the last term every time.

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