

Circuit Analysis II (AC)

University Electrical Engineering

What this course is, and how to use it

This is the fourth course in University Electrical Engineering: **AC circuit analysis** – circuits driven by sinusoids, which is how almost all electrical energy is generated and moved. Wave 1 solved DC; the tools of EE02 (complex numbers, differential equations, Laplace) now pay for themselves.

- Unit 4.1 makes sinusoids arithmetic: the **phasor** and the **impedance**.
- Units 4.2–4.4 apply it: AC power, three-phase, second-order response and resonance.
- Units 4.5–4.7 widen the view: filters and Bode plots, s-domain circuit analysis, and two-port networks.

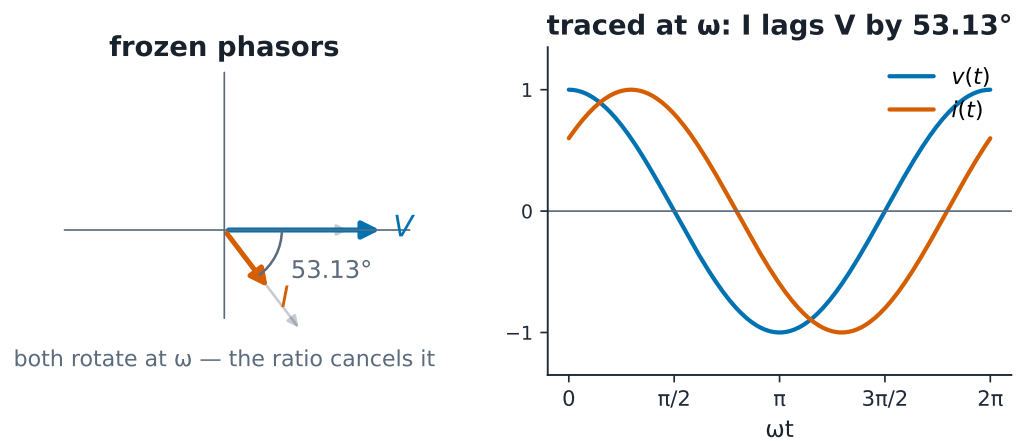
Each unit opens with a story, shows a picture before the formal idea, defines terms precisely, works two escalating examples whose numbers are machine-verified, names the classic mistakes, and closes with a three-line summary. The checking habit: every phasor answer verifies in a second representation – polar against rectangular, or the time domain against the complex arithmetic. Complex arithmetic is only bookkeeping; the checks keep it honest.

Sinusoidal steady state and phasors

The mains voltage crosses zero a hundred times a second, yet a resistor there dissipates steady power – because amplitude and phase settle into a repeating pattern. A sinusoid $v(t) = V_m \cos(\omega t + \varphi)$ becomes a **phasor** $V = V_m \angle \varphi$ – the amplitude and phase, with the frequency understood. Differentiating multiplies by $j\omega$; so at a single frequency every element gets an **impedance** :

$$Z_R = R \quad Z_L = j\omega L \quad Z_C = \frac{1}{j\omega C}$$

All the DC methods – dividers, node/mesh, Thévenin – carry over unchanged with complex numbers. The inductor's impedance grows with frequency ($+90^\circ$); the capacitor's shrinks (-90°); the resistor is flat (0°).



The rotating picture behind the frozen phasors: ratio questions cancel the rotation, so the snapshot is enough

Worked example 1 (both representations agree). $R = 30 \, \Omega$ in series with L such that $\omega L = 40 \, \Omega$, driven by $V = 100 \angle 0^\circ$ V at that frequency. $Z = 30 + j40 = 50 \angle 53.13^\circ \, \Omega$; $I = V/Z = (100/50) \angle (0 - 53.13^\circ) = 2 \angle -53.13^\circ$ A. Check in rectangular: $I = 100(30 - j40)/2500 = 1.2 - j1.6$ A; $|I| = \sqrt{1.44 + 2.56} = 2 \, \checkmark$ and $\text{atan2}(-1.6, 1.2) = -53.13^\circ \, \checkmark$. The time-domain meaning: $i(t) = 2 \cos(\omega t - 53.13^\circ)$ A – the current lags because the inductor opposes change.

Worked example 2 (parallel combination, escalated). The same $R = 30 \, \Omega$ now in **parallel** with the $\omega L = 40 \, \Omega$ inductor: $Z = \frac{30 \times j40}{30 + j40} = \frac{30 \times j40 \times (30 - j40)}{2500} = \frac{48000 + j36000}{2500} = 19.2 + j14.4 = 24 \angle 36.87^\circ \, \Omega$ (verified: $|Z| = \sqrt{19.2^2 + 14.4^2} = 24 \, \checkmark$, and $\cos 36.87^\circ = 0.8 \, \checkmark$). Note the parallel impedance is *smaller* than either branch magnitude ($24 < 30 < 40$) and its angle (36.87°) lies *below* the inductor's 90° , pulled toward the resistor – every parallel pair trades magnitude and angle in this direction.

Common mistakes. Adding magnitudes of parallel impedances (only complex addition is legal). Mixing degree-mode calculator settings into radian formulas. Forgetting that the phasor carries *amplitude* (or RMS – pick one convention per problem) consistently for V and I.

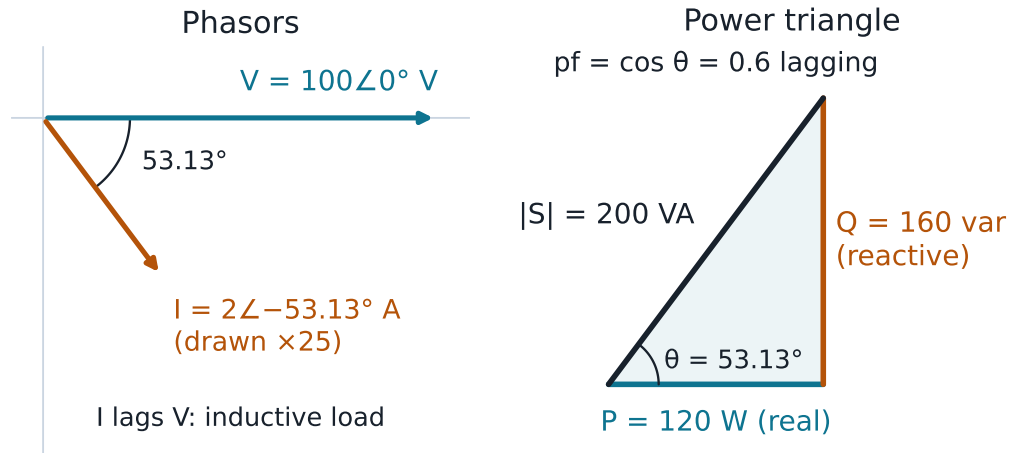
Summary: one frequency $\rightarrow$ phasors; Z_R flat, Z_L up 90° , Z_C down 90° ; every DC method carries over; verify each answer in the other representation.

AC power and three-phase

A 1 kW heater and a 1 kVA capacitor both draw “1 k-something” from the line, but only the heater’s share bills as energy. With phasor V and current I , **complex power** $S = V \cdot I^*$:

$$P = VI \cos \theta \quad (\text{real, W}), \quad Q = VI \sin \theta \quad (\text{reactive, var})$$

with θ the angle between V and I . The **power factor** $\cos \theta$ is the fraction of apparent power $|S| = VI$ that does real work.



The RL load: current lags by 53.13° , so $P = 120 \text{ W}$, $Q = 160 \text{ var}$ and $\text{pf} = 0.6$ lagging

Worked example 1 (the RL load, checked two ways). The load above: $P = 100 \times 2 \times \cos 53.13^\circ = 120 \text{ W}$, $Q = 100 \times 2 \times \sin 53.13^\circ = 160 \text{ var}$, $|S| = 200 \text{ VA}$, $\text{pf} = 0.6$ lagging. Check: $P = |I|^2 R = 4 \times 30 = 120 \checkmark$ and $Q = |I|^2 X = 4 \times 40 = 160 \checkmark$ – the loss-formula check. **Power-factor correction** adds parallel capacitance to supply the reactive power locally: to correct 160 var at 100 V (50 Hz) needs $C = Q/(\omega V^2) = 160/(2\pi \cdot 50 \cdot 10^4) = 50.9 \mu\text{F}$.

Worked example 2 (a three-phase motor, escalated). A balanced star-connected motor on a 415 V (line), 50 Hz supply has phase impedance $10 + j10 \Omega$ per phase. Phase voltage = $415/\sqrt{3} = 239.6 \text{ V}$; phase current = $239.6/\sqrt{10^2 + 10^2} = 239.6/14.14 = 16.94 \text{ A}$ (equal to line current in star); phase angle 45° lagging. Total power $P = 3 \times 239.6 \times 16.94 \times \cos 45^\circ = 8611 \text{ W}$ – and the line formula $\sqrt{3} \times 415 \times 16.94 \times 0.7071 = 8611 \text{ W} \checkmark$ the two routes agree. The per-phase route also gives the missing pieces: $Q = 3 \times 239.6 \times 16.94 \times \sin 45^\circ = 8611 \text{ var}$, pf 0.707 lagging, and (for a delta-connected motor of the same impedance) the phase current would be $415/14.14 = 29.3 \text{ A}$ with line current $\sqrt{3} \times 29.3 = 50.8 \text{ A}$ – the star/delta factor is exactly the $\sqrt{3}$ bookkeeping.

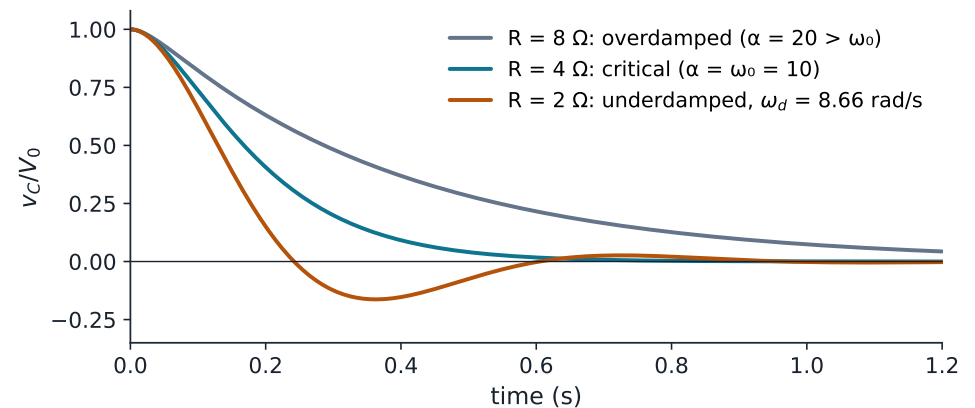
Common mistakes. Using line voltage in the per-phase formula (divide by $\sqrt{3}$ first). Forgetting the motor’s pf applies to *both* formulas. Correcting power factor at the wrong voltage – the capacitor sees the voltage at its own connection point.

Summary: $S = VI^*$ splits into P (billed) and Q (borrowed); loss formulas $|I|^2 R$, $|I|^2 X$ are the check; capacitive correction supplies Q locally; three-phase per-phase analysis with $V_{\text{line}} = \sqrt{3} V_{\text{phase}}$ (star) is the workhorse.

Second-order RLC circuits

Close a switch on a charged capacitor in series with an inductor and the current cannot jump – nor can it arrive instantly – so the response bends through at least one derivative more than RC ever needed. A series RLC has the differential equation $L d^2 i/dt^2 + R di/dt + i/C = 0$, with **natural frequency** $\omega_0 = 1/\sqrt{LC}$ and **damping** $\alpha = R/(2L)$. Three regimes:

- **Overdamped** ($\alpha > \omega_0$): two real decays, no oscillation.
- **Critically damped** ($\alpha = \omega_0$): fastest non-oscillating return.
- **Underdamped** ($\alpha < \omega_0$): decaying oscillation at $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.



One circuit, three resistors: the same $\omega_0 = 10 \text{ rad/s}$ with $\alpha = 20, 10$ and 5

Worked example 1 (regime arithmetic). $R = 4\ \Omega$, $L = 0.2\ \text{H}$, $C = 0.05\ \text{F}$: $\omega_0 = 1/\sqrt{0.2 \times 0.05} = 10\ \text{rad/s}$; $\alpha = 4/(2 \times 0.2) = 10$ – critically damped ($\alpha = \omega_0$). Change R to $2\ \Omega$: $\alpha = 5 < 10$, underdamped with $\omega_d = \sqrt{100 - 25} = 8.66\ \text{rad/s}$. The regime test is one comparison; the **Q factor** $Q = \omega_0/(2\alpha)$ is 0.5 at critical damping and 1 at $R = 2\ \Omega$ – lightly damped (high Q) rings longer.

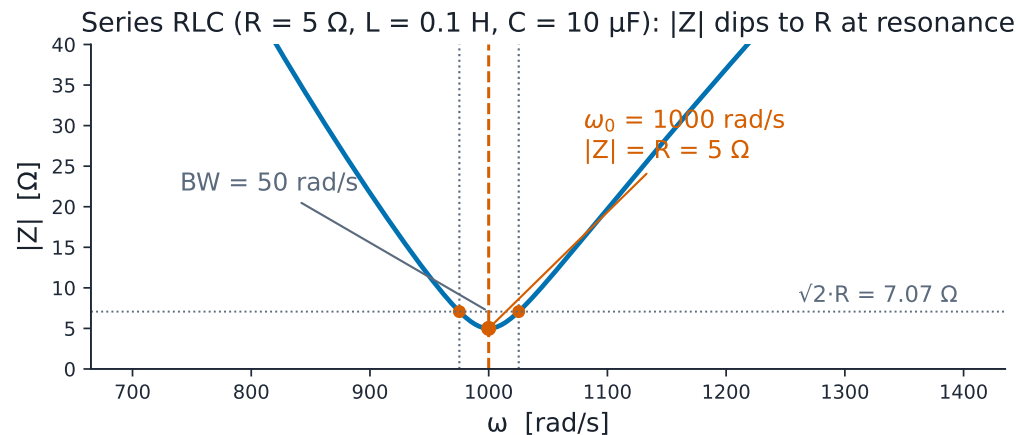
Worked example 2 (the overdamped exponents, machine-verified, escalated). With $R = 8\ \Omega$ (same L , C): $\alpha = 20 > 10$ – overdamped, and the two decay exponents are $s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -20 \pm \sqrt{300}$, i.e. -2.68 and $-37.32\ \text{s}^{-1}$ (machine-verified). The capacitor voltage released from 1 V is $v_C(t) = Ae^{-2.68t} + Be^{-37.32t}$ with $A + B = 1$ and $dv_C/dt(0) = 0$ (no instantaneous current), giving $A = 37.32/(37.32 - 2.68) = 1.077$, $B = -0.077$: a fast initial drop onto the slow decay. The reading: the *fast* root only sets the first moments; the *slow* root (-2.68 , the one nearer the origin) owns the tail – exactly the dominant-pole rule EE02 met in Laplace.

Common mistakes. Comparing R with $2\sqrt{L/C}$ instead of α with ω_0 – the same test, but write it as stated. Using ω_0 where ω_d belongs (the ringing you *see* is the damped frequency). Forgetting the second initial condition: two energy stores give two constants.

Summary: $\omega_0 = 1/\sqrt{LC}$, $\alpha = R/2L$, three regimes by one comparison; ω_d is the visible ringing; the slow root owns the tail; two stores $\rightarrow$ two constants $\rightarrow$ two endpoint checks.

Resonance

A radio's tuning circuit picks one station out of a crowded band by making one frequency special: at **resonance** the imaginary parts cancel, $\omega_0 L = 1/(\omega_0 C)$, i.e. $\omega_0 = 1/\sqrt{LC}$. There $Z = R$ (purely resistive, minimum), current is maximum, and the **bandwidth** $BW = R/L = \omega_0/Q$. The halves of the impedance cancel even though each alone can be large – V_L and V_C can each exceed the source voltage by the factor **Q**.



$|Z|$ dips to R at ω_0 ; the half-power points sit where $|Z| = \sqrt{2} \cdot R$

Worked example 1 (voltage magnification, checked). $R = 5\ \Omega$, $L = 0.1\ \text{H}$, $C = 10\ \mu\text{F}$: $\omega_0 = 1/\sqrt{0.1 \times 10^{-5}} = 1000\ \text{rad/s}$; $Q = \omega_0 L/R = 1000 \times 0.1/5 = 20$. At a source of 10 V, the current at resonance is $10/5 = 2\ \text{A}$, and $V_L = \omega_0 L I = 200\ \text{V}$ – twenty times the source. Check: $V_C = I/(\omega_0 C) = 2/0.01 = 200\ \text{V}$ ✓; both are 200 V, 180° apart, their sum zero.

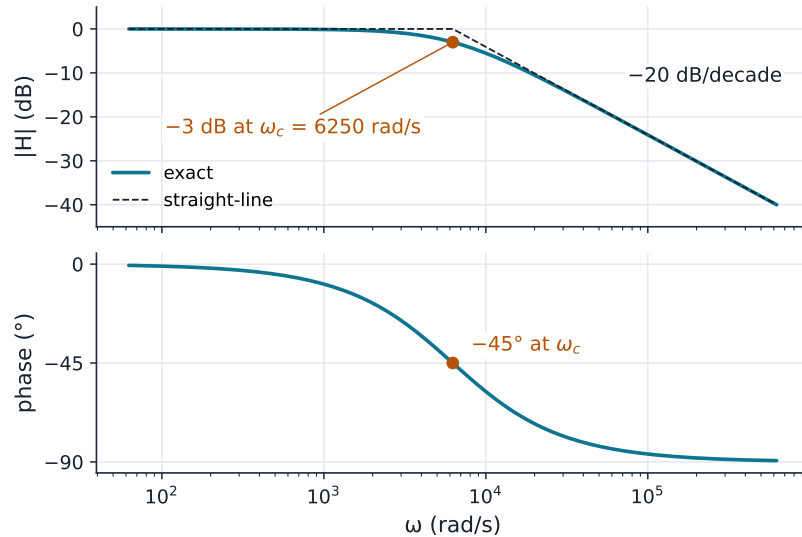
Worked example 2 (the half-power band, escalated). The same circuit's bandwidth: $BW = \omega_0/Q = 1000/20 = 50\ \text{rad/s}$, and the half-power frequencies compute to $\omega_1 = 975.3$, $\omega_2 = 1025.3\ \text{rad/s}$ – their difference is 50.0 ✓ (machine-verified). They sit almost evenly either side of 1000; exactly, $\omega_1 \omega_2 = \omega_0^2$ (symmetric on a logarithmic axis). At the band edges $|Z| = \sqrt{2}R = 7.07\ \Omega$, the current is $1/\sqrt{2}$ of maximum, and each reactive element holds $Q/\sqrt{2} \approx 14$ times the source voltage – still amplification, just past its peak. This is the selectivity a tuner sells: at 900 rad/s (a neighbouring station two bandwidths away) the current is down to $10/\sqrt{R^2 + (\omega L - 1/\omega C)^2} = 10/|5 + j(90 - 111.1)| = 10/21.7 = 0.46\ \text{A}$ – under a quarter of the resonant 2 A.

Common mistakes. Reporting Q -amplification without checking the other element (both must match). Using f_0 where ω_0 belongs (factor 2π). Believing resonance maximises *power* at any source impedance – it minimises $|Z|$, which maximises current only for a stiff source.

Summary: $\omega_0 = 1/\sqrt{LC}$, $|Z| = R$ there, $BW = \omega_0/Q$ between the half-power points; each reactive voltage is $Q \cdot \text{source}$; check the amplification on both elements.

Filters and Bode plots

A bass boost is a filter: one band up, the rest untouched. A **filter** passes a band and attenuates the rest; its transfer function $H(j\omega)$ has magnitude and phase. The **Bode plot** plots $20\log_{10}|H|$ in dB against log frequency: each pole contributes -20 dB/decade above its **corner frequency** $\omega_c = 1/(RC)$; each zero $+20$ dB/decade.



RC low-pass with its corner at 6250 rad/s: exact curves against the straight-line asymptotes

Worked example 1 (the low-pass, endpoints checked). A low-pass RC with $R = 1.6$ k Ω , $C = 0.1$ μ F: $\omega_c = 1/(1600 \times 10^{-7}) = 6250$ rad/s (995 Hz). At ω_c the gain is $1/\sqrt{2} = -3$ dB and the phase -45° ; at $10\omega_c$ the straight-line approximation reads -20 dB (exact -20.04). Check the endpoints: at low frequency $H = 1$ (0 dB), at $100\omega_c \approx -40$ dB – each decade past the corner loses 20 dB, which is the **decibel** arithmetic $20\log_{10} 10 = 20$.

Worked example 2 (the high-pass mirror, machine-verified, escalated). Swapping output to the resistor makes a high-pass with the same $\omega_c = 6250$ rad/s: $H(j\omega) = (j\omega/\omega_c)/(1 + j\omega/\omega_c)$. At $0.1\omega_c$: -20.04 dB at $+84.3^\circ$; at ω_c : -3.01 dB at $+45^\circ$; at $10\omega_c$: -0.04 dB at $+5.7^\circ$ (all machine-verified) – the exact mirror of the low-pass, with phase *leading* and the stop-band below instead of above. The design reading: a steeper 40 dB/decade roll-off needs two cascaded sections, and (with the loading between them accounted for) the cascade multiplies gains, so the dB add – the same chaining logic as the ABCD matrices of unit 4.7.

Common mistakes. Reading -3 dB as “half power lost in the filter” at the corner

(it is half power *delivered to the load versus mid-band*). Mixing ω and f on the same axis. Forgetting phase: a filter that is flat in magnitude can still wreck a pulse by its phase slope.

Summary: poles -20 dB/decade above their corner, zeros $+20$; -3 dB and $\pm 45^\circ$ at the corner; the asymptotes are the design tool, the exact curve the check.

Laplace circuit analysis

EE02’s Laplace machinery becomes a circuit method: each element gets an s-domain model with initial conditions as sources –

- Resistor: $Z = R$.
- Inductor: $Z = sL$ **plus a series voltage** $Li(0^-)$ (or parallel current source $i(0^-)/s$).
- Capacitor: $Z = 1/(sC)$ **plus a parallel current source** $Cv(0^-)$ (or series voltage $v(0^-)/s$).

Then every DC method applies in s . The response’s **poles** carry the transient; the input’s poles carry the steady state.

The s-domain models: impedance plus initial-condition source

Resistor $Z = R$ no initial energy — nothing extra

Inductor $Z = sL$ plus series source $Li(0^-)$

Capacitor $Z = 1/(sC)$ plus parallel source $Cv(0^-)$

Then every DC method (divider, mesh, Thévenin) applies in s — with the sources carrying the past

The s-domain models: impedance plus initial-condition source – one picture for every transient problem

Worked example 1 (the wave-1 RC, by algebra). The wave-1 RC step with initial voltage: $R = 10$ k Ω , $C = 100$ μ F ($\tau = 1$ s), source 9 V, $v_C(0) = 1$ V. s-domain KVL: $9/s = IR + v_C(0)/s + I/(sC) \rightarrow I = 8/(R(s + 1/(RC))) = 8/(10^4(s + 1))$; $V_C = v_C(0)/s + I/(sC) = 1/s + 8/(s(s + 1)) = 9/s - 8/(s + 1)$, so $v_C(t) = 9 - 8e^{-t}$ V – exactly the wave-1 answer, now derived by algebra. The check: $v_C(0) = 9 - 8 = 1$ ✓, $v_C(\infty) = 9$ ✓.

Worked example 2 (a source-free RLC, poles matched, escalated). The series RLC of unit 4.3 with $R = 2\ \Omega$, $L = 0.2\ \text{H}$, $C = 0.05\ \text{F}$, released with $i(0) = 1\ \text{A}$, $v_C(0) = 0$. s-domain mesh (impedances plus the inductor's $Li(0)$ source): $(0.2s + 2 + 20/s)I = 0.2$, so $I(s) = s/(s^2 + 10s + 100)$ and the poles are $-5 \pm j8.66$ – exactly the $\alpha = 5$, $\omega_d = 8.66$ the time-domain regime arithmetic gave ✓. The two routes are the same physics in two notations; the s-domain route delivers the exponents without memorising a form, and partial fractions (as in EE02) finish the waveform.

Common mistakes. Dropping the initial-condition sources – the answer then claims the circuit starts at rest. Cancelling s blindly (a factor of s in numerator and denominator can hide an impulse). Forgetting the final-value theorem's condition (pole strictly in the left half-plane).
Summary: impedances become R , sL , $1/sC$ plus initial-condition sources; DC methods in s ; the poles *are* α and ω_d ; endpoint checks still free.

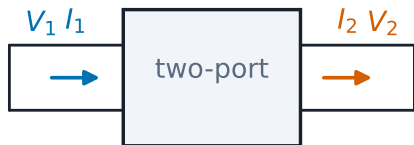
Two-port networks

An amplifier module, a matching network, a length of cable – each is a box with two wire pairs, and the designer only needs what the box does at its terminals. A **two-port** has input and output terminal pairs; the outside world sees only the port behaviour. Parameter sets relate the port variables:

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = [ABCD] \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix}, \quad Z: \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = [z] \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

Conversion is algebra (e.g. from z-parameters: $A = z_{11}/z_{21}$, $B = (z_{11}z_{22} - z_{12}z_{21})/z_{21}$).
Cascade connections multiply ABCD matrices – the reason transmission lines and filter chains use this set.

port conventions



ABCD: current OUT of port 2

elementary ABCD blocks

series Z $(1, Z; 0, 1)$
impedance in the line

shunt Y $(1, 0; Y, 1)$
admittance across

cascade = matrix multiply

Port conventions and the two elementary ABCD blocks every chain composes

Worked example 1 (the L-section cascade, determinant checked). The series impedance Z has $ABCD = (1, Z; 0, 1)$; the shunt admittance Y has $(1, 0; Y, 1)$. Cascading shunt- Y then series- Z : $(1, 0; Y, 1)(1, Z; 0, 1) = (1, Z; Y, 1 + YZ)$. Check with $Z = j10$, $Y = -j0.1$ (an L section at one frequency): $1 + YZ = 1 + (-j0.1)(j10) = 2$, $B = j10$ – and the determinant $AD - BC = 2 - j10 \cdot (-j0.1) = 1$ equals 1, as every ABCD matrix of a reciprocal network must.

Worked example 2 (into z-parameters, machine-verified, escalated). The same L-section's z-parameters from the conversion $z_{11} = A/C$, $z_{12} = \Delta/C$, $z_{21} = 1/C$, $z_{22} = D/C$ with $C = Y = -j0.1$: $z_{11} = z_{12} = z_{21} = 1/(-j0.1) = j10\ \Omega$, $z_{22} = 2/(-j0.1) = j20\ \Omega$ (machine-verified). The symmetry $z_{12} = z_{21}$ is the reciprocity condition – visible in the numbers, not assumed – and $z_{22} \neq z_{11}$ records the asymmetry of the L (shunt at the input side). Read back: with port 2 open ($I_2 = 0$), $V_1 = j10I_1$ – the series arm carries no current, so the input sees only the shunt arm, $1/Y = 1/(-j0.1) = j10\ \Omega$.

Common mistakes. Sign of I_2 : ABCD uses $-I_2$ (current *out* of port 2); z-parameters use it *into* the port – pick the convention and stay. Multiplying cascades in the wrong order (matrix product is not commutative). Expecting $z_{12} = z_{21}$ for a network with controlled sources inside – reciprocity requires linearity *without* dependent sources.

Summary: ports hide the interior; ABCD cascades by multiplication (reciprocal networks: $AD - BC = 1$); z-parameters suit series connections and expose reciprocity as $z_{12} = z_{21}$; convert only by algebra, verify by one determinant.