Mathematical Methods for EE

University Electrical Engineering

What this course is, and how to use it

This is the second course in University Electrical Engineering: **mathematical methods** – the toolkit the rest of the degree runs on. Wave 1's DC circuits needed only algebra; from AC circuits onward, every method rests on something taught here.

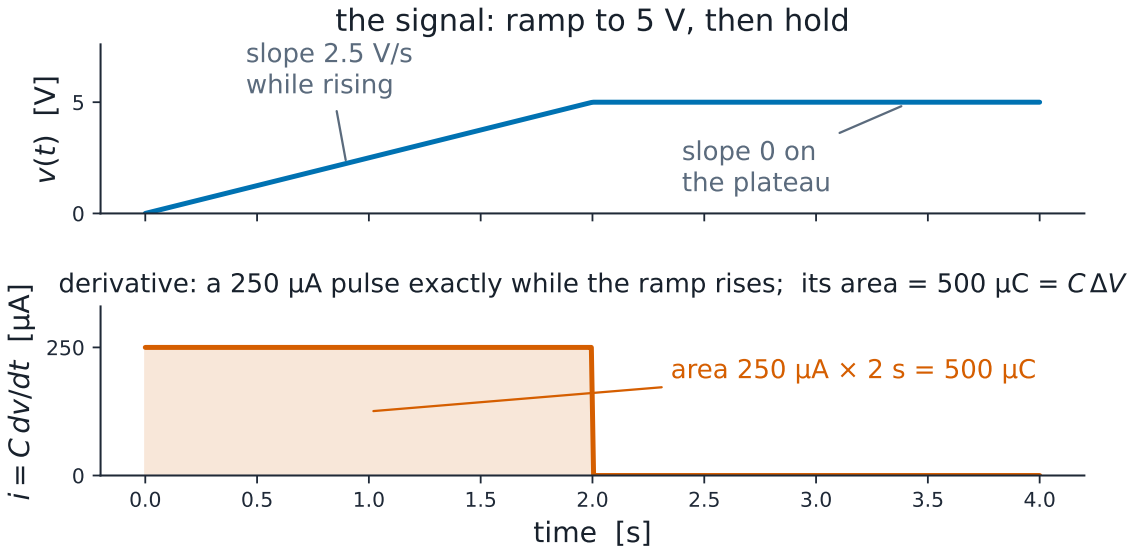
- Units 2.1–2.3 refresh and extend the algebra: calculus for signals, complex numbers, linear algebra.
- Units 2.4–2.6 are the dynamics toolkit: differential equations, Laplace transforms, Fourier series.
- Units 2.7–2.8 handle randomness and fields: probability for noise, vector calculus for electromagnetics.

Each unit opens with a story, shows a picture before the formal idea, defines terms precisely, derives what later courses quote, works two escalating examples whose numbers are machine-verified, names the classic mistakes, and closes with a three-line summary. The habit is the same one the whole degree uses: check every symbolic answer numerically at a concrete value – an integral by a sum, a solved system by substitution, a transform pair by computing both sides.

Calculus review for signals

A scope trace rises smoothly from 0 to 5 V and flattens. The current into a capacitor at every instant depends on *how fast* the trace moves – slope – and the charge delivered over an interval is the *area* under the current trace. Signals are functions of time, and calculus reads them: the **derivative** dv/dt is the slope – instantaneous rate of change; the **integral** $\int i\,dt$ is accumulated area.

$$\frac{d}{dt}t^n = n\,t^{n-1} \qquad \int_0^T t^n\,dt = \frac{T^{n+1}}{n+1}$$



The same signal, read twice: slope on top, area below

For the EE work ahead, three facts do most of the work:

- The derivative of e^{at} is $a \cdot e^{at}$ – exponential rates are their own slopes.
- The integral of a **rectangular pulse** (height A, width τ) is area $A\tau$.
- A triangle's area is half the enclosing rectangle – and $\int_0^\infty Ae^{-t/\tau}\,dt = A\tau$ (the exponential decays over its time constant, so the "effective width" is τ).

Deriving e^{at}'s self-slope. The limit definition: $\frac{d}{dt}e^{at} = \lim_{h\to 0}\frac{e^{a(t+h)}-e^{at}}{h} = e^{at}\lim_{h\to 0}\frac{e^{ah}-1}{h} = e^{at}\cdot a$ – the exponential factors out of its own difference quotient, which is the whole reason circuits that vary as e^{at} stay that way under differentiation.

> **Worked example 1 (the linear ramp, verified).** A capacitor charging linearly from 0 to 5 V in 2 s ($v = 2.5t$): with $C = 100\ \mu F$, $i = C\,dv/dt = 100\mu \times 2.5 = 250\ \mu A$ – constant, because the ramp's slope is constant. The charge delivered is $\int i\,dt = 250\mu \times 2 = 500\ \mu C$, which also equals $C \times \Delta V = 100\mu \times 5$. Two methods, one answer – that is the checking habit.

> **Worked example 2 (the exponential pulse, escalated).** A sensor pulse decays as $i(t) = 5e^{-t/0.2}$ mA (peak 5 mA, time constant $\tau = 0.2$ s). The total charge is the area $\int_0^\infty 5e^{-t/0.2}dt = 5 \times 0.2 = 1.0$ mC – the "effective width" rule in action. The peak slope is at $t = 0$: $\frac{di}{dt} = -\frac{5}{0.2} = -25$ mA/s. Check by crude summation: one rectangle of height 5 mA over the width 0.2 s gives exactly the same 1.0 mC – which is not a coincidence but the meaning of τ.

Common mistakes. Differentiating a ramp's *value* instead of its slope (constant 5 V does not mean constant current – only constant *slope* does). Forgetting that integration is signed: area below the axis subtracts. Dropping units inside the integral ($\mu A \times s = \mu C$, always).

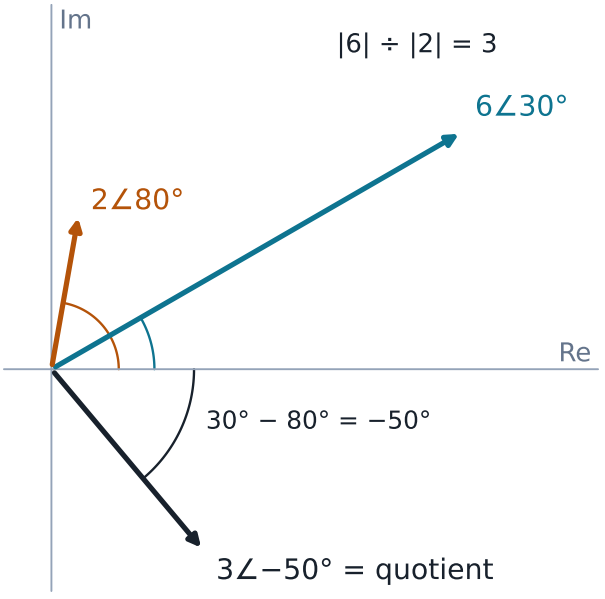
Summary: derivative = slope, integral = signed area; exponentials are their own slopes; a pulse's area is height × effective width (τ for a decaying exponential) – and every answer has a unit.

Complex numbers and phasor algebra

An AC voltage is 230 V *and* some phase – one number cannot hold both, but a point on a plane can. A **complex number** $z = a + jb$ (with $j^2 = -1$) has a real part, an imaginary part, a **modulus** $|z| = \sqrt{a^2 + b^2}$ and an **argument** $\theta = \operatorname{atan2}(b, a)$. The two forms are interchangeable:

$$a + jb = r(\cos\theta + j\sin\theta) = r\,e^{j\theta}$$

– **Euler's identity** , the bridge EE uses everywhere. Multiplication adds arguments: $r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$ – a rotation and a scaling, which is exactly what a sinusoid's phase shift and amplitude change look like.



Division in polar form: 6 ÷ 2 = 3 and 30° − 80° = −50°

Worked example 1 (divide two phasors, checked by the inverse). Divide $6\angle 30°$ by $2\angle 80°$: in polar form the answer is $3\angle -50°$ directly – moduli divide, arguments subtract. Doing the same in rectangular form means rationalising $(5.196 + j3)/(0.347 + j1.970)$ – possible, but the polar route is why EE works in **phasors** . A third check: $3\angle -50° \times 2\angle 80° = 6\angle 30°$ – the inverse operation must return the start.

Worked example 2 (a quadratic with complex roots, escalated). A damped system's characteristic equation $s^2 + 2s + 5 = 0$ has roots $s = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm j2$. In polar form the roots are the conjugate pair $2.236\angle \pm 116.6°$ (modulus $\sqrt{1+4} = 2.236$). Check by substitution: $(-1+j2)^2+2(-1+j2)+5 = (-3-j4)+(-2+j4)+5 = 0$ exactly. The roots' real part -1 is the decay rate and the imaginary part 2 rad/s the ringing frequency – one complex number carrying both behaviours, which is why the rest of the degree keeps them together.

Common mistakes. Adding moduli instead of adding complex numbers (moduli do not add; rectangular parts do). Using degrees inside $e^{j\theta}$ formulas that assume radians. Confusing j rotation direction: multiplying by j rotates $+90°$, by $-j$ rotates $-90°$.

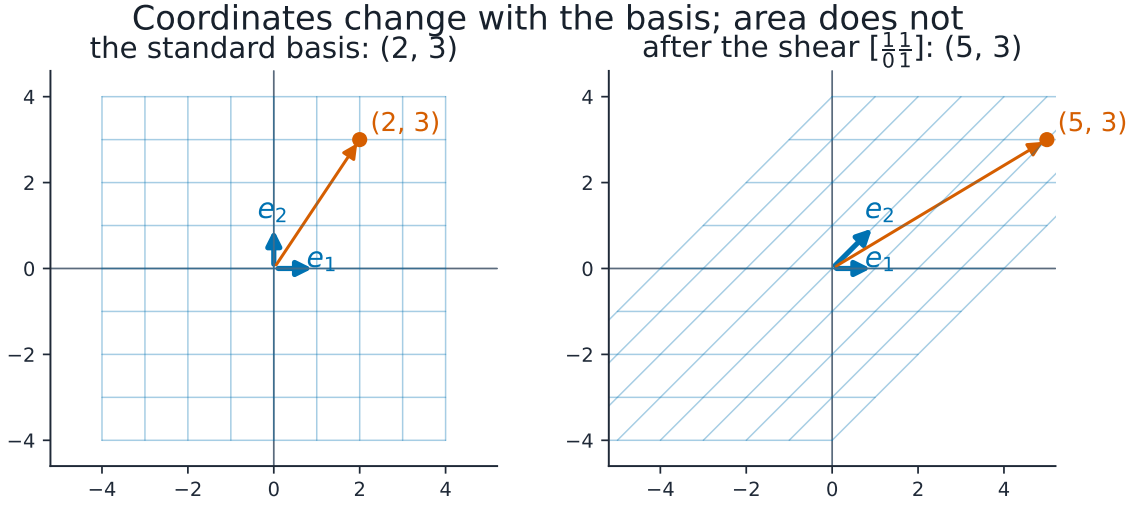
Summary: two forms, one number – rectangular for adding, polar for multiplying and dividing; Euler's identity is the bridge; roots of quadratics carry decay in the real part and frequency in the imaginary part.

Linear algebra essentials

Three mesh currents in a coupled circuit mean three equations in three unknowns – and the structure of the solution is not "clever substitution" but a matrix. A **linear system** $Ax = b$ is solved by elimination; a **matrix inverse** A^{-1} exists only when the **determinant** is non-zero. For the 2×2 case:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The determinant $ad - bc = 0$ means the equations are **singular** – no unique solution, and in circuits a dependent source loop or a floating node.



A basis spans the space; a matrix moves between bases

Deriving the 2×2 inverse. Multiply A by its candidate: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = I$ – the swap-and-negate pattern exists precisely because those cross-terms cancel.

> **Worked example 1 (elimination and inverse agree).** Solve $x+2y=8$, $3x+y=9$. Elimination: subtract 3×row1 from row2 → $-5y=-15$, so $y=3$, $x=2$. Check by the inverse: $\det = 1\cdot1-2\cdot3=-5$; $A^{-1} = \frac{1}{-5}\begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$; $x=(1\cdot8-2\cdot9)/-5=2$ ✓, $y=(-3\cdot8+1\cdot9)/-5=3$ ✓. Both routes agree; each is the other's check.

> **Worked example 2 (the mesh system, machine-solved, escalated).** The EE01 two-mesh circuit closes as one matrix equation $\begin{pmatrix} 6 & -4 \\ -4 & 10 \end{pmatrix}\begin{pmatrix} i_A \\ i_B \end{pmatrix} = \begin{pmatrix} 12 \\ -3 \end{pmatrix}$. Elimination (or the inverse; $\det = 60-16=44$) gives $i_A = 2.455$ A, $i_B = 0.682$ A – verified by substitution into both original equations to machine precision. The negative off-diagonal entries -4 are the shared resistor speaking: coupling shows up as off-diagonal terms, and the bigger the coupling, the further the determinant sits from the product of the diagonals. Now the singular case: $x+2y=8$, $2x+4y=16$ has determinant $1\cdot4-2\cdot2=0$ – the second equation is just twice the first (a whole line of solutions, not one point).

Common mistakes. Mixing the row order between A and b. Assuming a non-zero determinant guarantees an easy solution – it guarantees uniqueness, not convenience. Forgetting that singular systems can have *no* solution (parallel planes) or *infinitely many* (dependent planes): the determinant cannot tell which.

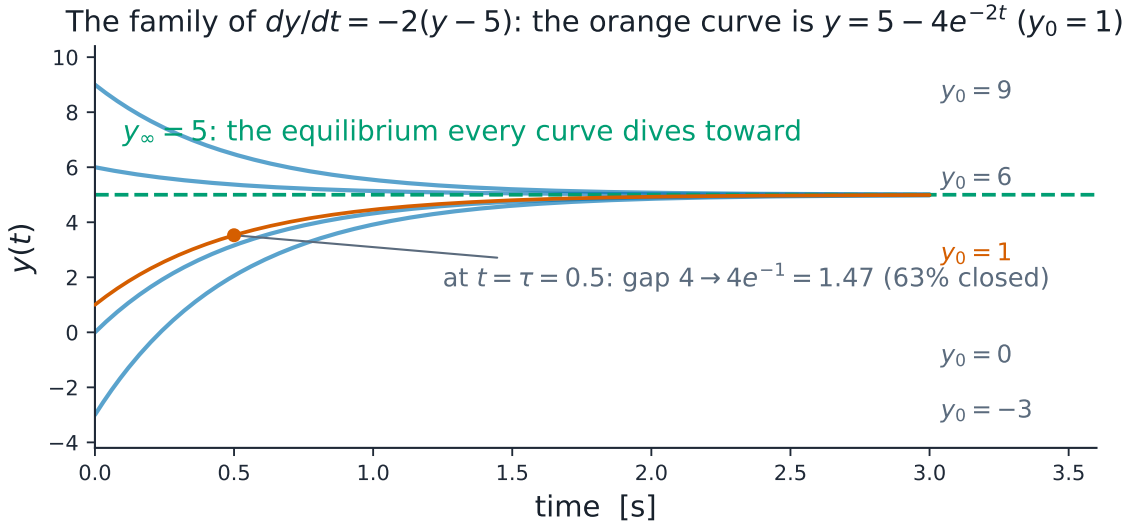
Summary: $Ax=b$ with $\det \neq 0$ has exactly one solution; elimination computes it, the inverse re-derives it, and substitution checks it; coupling lives in the off-diagonals; $\det = 0$ means the geometry collapsed.

Ordinary differential equations

The switch closes at $t=0$ and the capacitor voltage slides to its new value – wave 1's RC response was one instance of a pattern this unit makes general. The circuit equations are **ordinary differential equations** . A first-order linear ODE with constant coefficients,

$$\frac{dy}{dt} + ay = f(t),$$

has solution = **homogeneous** part Ce^{-at} + **particular** part forced by f. The homogeneous term carries the **initial condition** .



One equation, a family of curves; the initial condition selects one

Why the solution has this shape. For $\frac{dy}{dt} = -a(y - y_\infty)$, rewrite as $\frac{dy}{y-y_\infty} = -a\,dt$: each side is now integrable, giving $\ln(y - y_\infty) = -at + C$, i.e. $y - y_\infty = \tilde{C}e^{-at}$. The gap to the final value decays exponentially – every first-order system slides down its own percentage curve.

Worked example 1 (the step-solved RC, endpoint-checked). $dy/dt + 2y = 10$, $y(0) = 1$. Homogeneous: Ce^{-2t}. Particular (constant input → constant output): $y_p = 10/2 = 5$. So $y = 5 + Ce^{-2t}$; $y(0) = 5 + C = 1$ gives $C = -4$: $y = 5 - 4e^{-2t}$. Checks: at $t = 0$, $y = 1$ ✓; as $t \to \infty$, $y \to 5$ and $dy/dt \to 0$, so the equation reads $0 + 2(5) = 10$ ✓. This is the RC equation of unit 1.7 in general form – the method, not the circuit, is what this course teaches.

Worked example 2 (a decaying input, machine-verified, escalated). Solve $\frac{dy}{dt} + 3y = 6e^{-t}$, $y(0) = 2$. Try a particular $y_p = ke^{-t}$: $-ke^{-t} + 3ke^{-t} = 2ke^{-t} = 6e^{-t}$ gives $k = 3$. So $y = 3e^{-t} + Ce^{-3t}$; $y(0) = 3 + C = 2$ gives $C = -1$: $y = 3e^{-t} - e^{-3t}$. Check at $t = 0.5$: $y = 3(0.6065) - 0.2231 = 1.596$. Substituting into the left side: $dy/dt + 3y = -1.150 + 3(1.596) = 3.639$, and the right side is $6e^{-0.5} = 3.639$ – equal to machine precision. The response follows two exponentials, the input's (-1) and the system's (-3).

Common mistakes. Solving the homogeneous equation and forgetting the particular part entirely. Letting the initial condition choose k in the *particular* guess (only the homogeneous constant is free). Checking only $t = 0$: also check the large-t limit, where the equation must balance with $dy/dt \approx 0$.

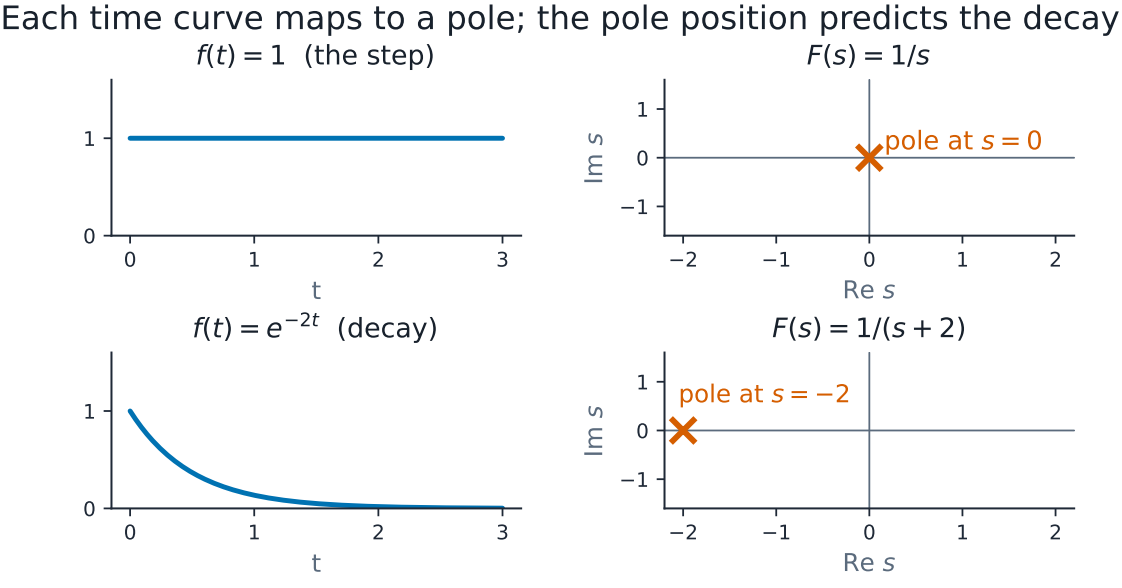
Summary: homogeneous + particular; the initial condition selects from the family; separation of variables explains the exponential shape; endpoint checks at both ends are free.

Laplace transforms

Solving ODEs by hand works for one equation; a circuit with three storage elements needs three. The **Laplace transform** turns differentiation into multiplication by s, converting entire systems into algebra:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt, \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Deriving the key pair. $\int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}$ (for $\operatorname{Re} s > a$) – the exponential eats its own decay, leaving a simple pole. That single computation generates most of the table.



Each time curve maps to a pole; pole position predicts decay

Three working rules: $\mathcal{L}\{dy/dt\} = sY - y(0)$ (the initial condition enters algebraically); $\mathcal{L}\{1\} = 1/s$; and **partial fractions** split a rational $F(s)$ into invertible pieces.

> **Worked example 1 (the unit-2.4 ODE, by transform).** Solve $dy/dt + 2y = 10$, $y(0) = 1$. Transformed: $sY - 1 + 2Y = 10/s$, so $Y = (10/s + 1)/(s + 2) = 10/[s(s + 2)] + 1/(s + 2)$. Partial fractions: $10/[s(s + 2)] = 5/s - 5/(s + 2)$. So $Y = 5/s - 4/(s + 2)$, and inverting term by term: $y = 5 - 4e^{-2t}$ – identical to the classical route, with the initial condition handled by bookkeeping instead of algebra at the end. The **poles** of Y (here $s = 0, -2$) predict the behaviour: a constant and a decaying exponential.

Worked example 2 (second order, machine-verified, escalated). $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$, $y(0) = 1$, $y'(0) = 0$. Transform: $(s^2 + 3s + 2)Y = s + 3$ (the derivative rule applied twice), so $Y = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$. Invert: $y = 2e^{-t} - e^{-2t}$. Checks: $y(0) = 2 - 1 = 1$ ✓; $y'(0) = -2 + 2 = 0$ ✓; $y(1) = 0.736 - 0.135 = 0.600$ (machine-verified). The two poles $-1, -2$ *are* the two decays, with the slower pole -1 dominating the tail – the observation that later courses turn into dominant-pole approximations.

Common mistakes. Forgetting the $-y(0)$ in the derivative rule (and its second-order cousins $-y'(0) - s\,y(0)$). Inverting a Y whose fractions are not fully split: $\frac{s+3}{(s+1)(s+2)}$ must become two simple pieces before any table lookup. Reading a zero at the numerator as a pole (a zero blocks; a pole decays).

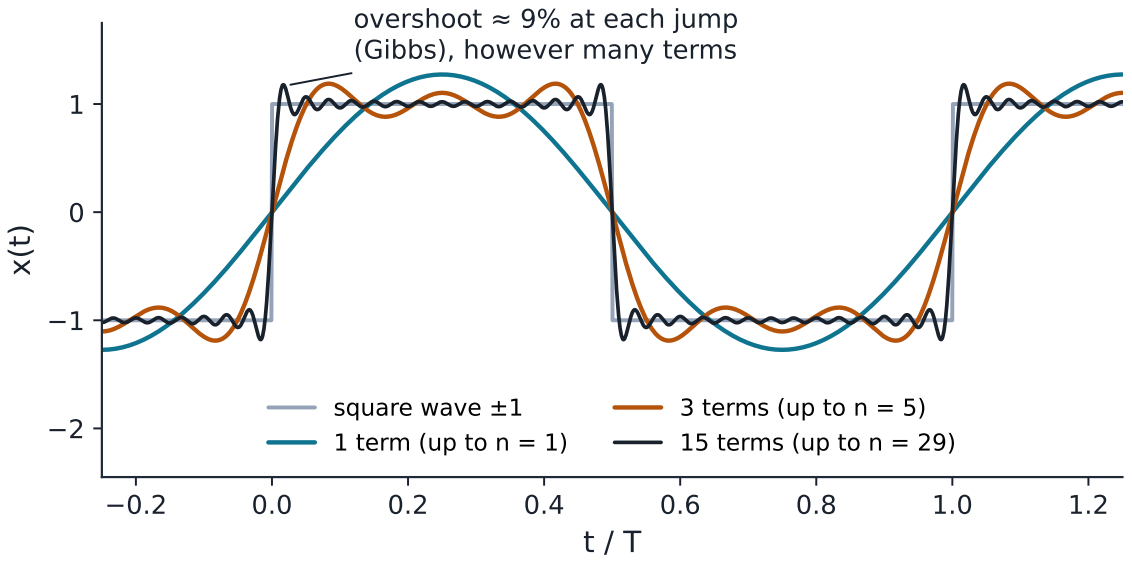
Summary: differentiation → multiplication by s, initials included; partial fractions split Y; poles predict the time behaviour; the transform and classical routes must agree – that agreement is the check.

Fourier series

A distorted mains waveform heats a resistor exactly as some combination of pure sinusoids would – and which combination is not a guess but a computation. Any periodic signal is a sum of sinusoids. For period T, **fundamental** $\omega_0 = 2\pi/T$, the **Fourier series** is

$$f(t) = a_0 + \sum_{n\geq 1} \big(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\big), \qquad a_0 = \frac{1}{T}\int_T f\,dt$$

with each a_n, b_n an average of f against the matching sinusoid (**orthogonality** : distinct sinusoids average to zero against each other – so each coefficient can be picked out independently, like a bank of tuned filters).



Odd harmonics $4/(n\pi)$ build the square wave; the Gibbs overshoot never goes away

Worked example 1 (the square wave, verified at the magic point). A square wave, ± 1, period T, odd symmetry: all $a_n = 0$, and $b_n = 4/(n\pi)$ for odd n – the series $\frac{4}{\pi}(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \dots)$. The check that convinces: at $t = T/4$ the sum must give 1, and $\frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \dots)$ is the Gregory series for $\pi/4$ scaled – it converges to 1, exactly as the square wave demands. Every **harmonic** amplitude falls as $1/n$: sharper edges need more harmonics, which is the whole of filtering.

Worked example 2 (power in the harmonics, machine-verified, escalated). A ± 1 V square wave across 1 Ω delivers average power 1 W (the mean of a constant magnitude). Parseval says the harmonic powers must sum to the same 1 W: $\sum_{n\ \text{odd}} \frac{1}{2}\left(\frac{4}{n\pi}\right)^2 = \frac{8}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right)$. Summing the first 20 odd harmonics gives 0.990 W; the full series converges to 1 W exactly (the sum of reciprocal odd squares is $\pi^2/8$, making the total $\frac{8}{\pi^2}\cdot\frac{\pi^2}{8} = 1$). The practical reading: the fundamental alone carries $\frac{8}{\pi^2} \approx 81\%$ of the power – a low-pass filter that keeps only the first harmonic still delivers most of the energy, and the 1/3-amplitude third harmonic adds just 9%.

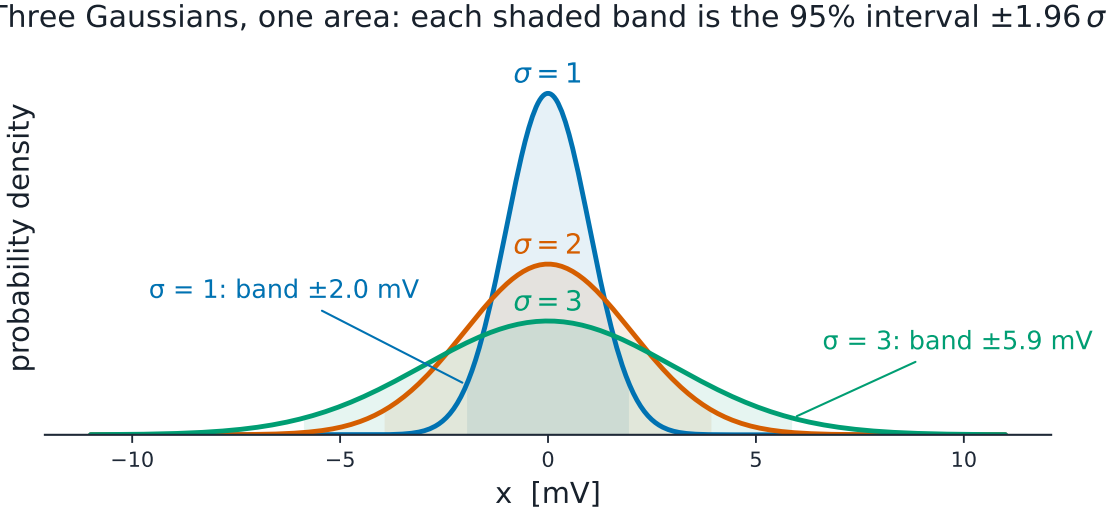
Common mistakes. Using $1/T$ (hertz) instead of $\omega_0 = 2\pi/T$ (rad/s) as the fundamental frequency. Forgetting the $\frac{1}{2}$ in each harmonic's power (a sinusoid's mean square is half its peak square). Assuming even symmetry everywhere: shift the origin and coefficients change – the spectrum belongs to the function as placed.

Summary: periodic signals decompose into harmonics; orthogonality picks each coefficient independently; amplitudes fall as $1/n$ for edges; Parseval audits the power; the Gibbs overshoot never leaves.

Probability and random variables

Every measurement is noisy; the noise itself is a signal you cannot predict – but its *statistics* are stable, and that stability is what instruments exploit. A **random variable** has a **probability distribution**; the Gaussian (normal) $N(\mu, \sigma^2)$ models thermal noise:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$



Same area, different spread: σ sets the width

Expectation $E[X]$ is the long-run average; **variance** $Var(X) = E[X^2] - \mu^2$ is the spread, and σ – the **standard deviation** – is the noise amplitude an oscilloscope shows. Two rules do most of the work: independent variances *add* ($\sigma_{total} = \sqrt{\sigma_1^2 + \sigma_2^2}$), and the average of n independent samples has standard deviation $\sigma/\sqrt{n}$.

> **Worked example 1 (the 2σ rule and averaging).** For $X \sim N(0, \sigma^2)$ with $\sigma = 2$ mV: $E[X] = 0$, $Var = 4$ (mV)^2, $\sigma = 2$ mV. The chance of exceeding $\pm 2\sigma$ is 4.55% (the two-tailed 95% band is $\pm 1.96\sigma \approx \pm 3.9$ mV). Averaging 16 independent samples yields $2/\sqrt{16} = 0.5$ mV of noise – the entire rationale of instrument averaging.

Worked example 2 (two noise sources, dB accounting, escalated). An amplifier stage adds 2 mV of noise to a signal that already carries 3 mV from the previous stage. The combined noise is not $3+2=5$ mV but $\sqrt{3^2+2^2}=\sqrt{13}=3.61$ mV – variances add, not deviations, so the smaller source contributes proportionally less than its face value. Signal-to-noise improvement from averaging those same 16 readings: $20\log_{10}(4)=12.0$ dB. The design reading: sixteen readings buy 12 dB, and each further 6 dB costs four times as many readings (and four times the measurement time); fixing the dominant noise source at its origin costs no measurement time – halving the 3 mV to 1.5 mV would give $\sqrt{1.5^2+2^2}=2.5$ mV, a 3.2 dB improvement on every single reading.

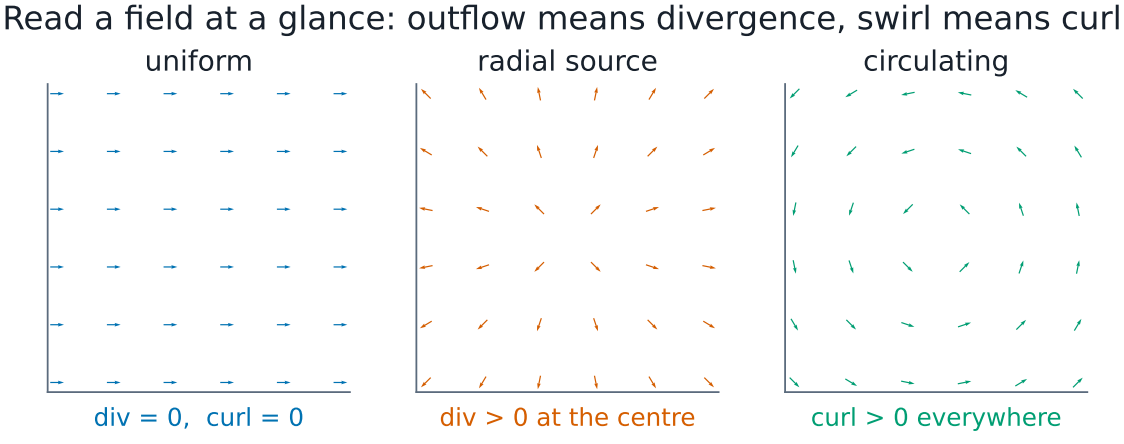
Common mistakes. Adding standard deviations instead of variances. Confusing σ and σ^2 in tables and formulas (the scope shows σ; formulas usually use σ^2). Forgetting that $\sigma/\sqrt{n}$ improvement assumes independence – correlated noise (mains hum) does not average away.

Summary: the Gaussian is the noise model; σ is the scope's noise; independent variances add, averaging divides σ by $\sqrt{n}$; attack the dominant source, then average.

Vector calculus for fields

A voltage probe reports one number at each point in space – a whole landscape of values, and EE07's Maxwell equations are statements about that landscape's geometry. Fields are vector-valued functions of position. Three operators on a field $\mathbf{F}$:

- **Gradient** ∇V: points up the steepest slope of a scalar; the electric field is $E=-\nabla V$.
- **Divergence** $\nabla\cdot\mathbf{F}$: the net outflow per unit volume – charge density for electric fields.
- **Curl** $\nabla\times\mathbf{F}$: the circulation per unit area – zero for electrostatics, the mechanism of induction where it is not.



Read a field's story at a glance: outflow means divergence, swirl means curl

Worked example 1 (the gradient predicts the drop, verified numerically). $V(x,y) = 3x^2y$ volts. $\nabla V = (6xy, 3x^2)$; at the point $(1,2)$: $\nabla V = (12,3)$, so $E = -(12,3)$ V/m. The check: move a small step $\delta = (0.01, 0)$ from $(1,2)$ – V changes by $\approx 6xy \cdot \delta x = 12 \times 0.01 = 0.12$ V, and $-E \cdot \delta = 12 \times 0.01 = 0.12$ V: the field times the step predicts the potential drop, which is what gradient means.

Worked example 2 (divergence and curl by hand, escalated). For $\mathbf{F} = (x^2,\ xy,\ 3x^2z)$: $\nabla \cdot \mathbf{F} = 2x + x + 3x^2$, which at $(1,1,1)$ equals $2 + 1 + 3 = 6$ – a net source. For the circulating field $\mathbf{G} = (-y,\ x,\ 0)$: the z-component of curl is $\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} = 1 - (-1) = 2$ – uniform rotation of strength 2 everywhere: the velocity field of a rigid body spinning at 1 rad/s, whose curl is twice its angular velocity. These two computations are the pointwise versions of EE07's Gauss and Stokes laws.

Common mistakes. Taking the gradient of a vector or the divergence of a scalar – gradient is for scalars, divergence for vectors. Sign slips in curl: the z-component is $\partial F_y/\partial x - \partial F_x/\partial y$; reversing them flips the rotation. A uniform field has zero divergence AND zero curl.

Summary: gradient is steepest-up (scalar→vector), divergence is net outflow (vector→scalar), curl is circulation (vector→vector); a step must reconcile $E \cdot \delta$ with δV.