7.1 Vector field tools

Name: _____ Class: _____ Date: _____ **Total: 21 marks**

KEY WORDS flux

Objective

Choose coordinates by symmetry, compute divergence/curl/flux integrals, and verify every integral-theorem result by a direct surface or line check.

1 Key knowledge and worked examples

The symmetry method

- Spherical symmetry → spheres $dA = r^2 sin\theta d\theta d\varphi$.
- Cylindrical → coaxial cylinders $dA = rd\varphi dl$.
- Planar → pillbox caps.
- No symmetry → direct integration.

Worked example: divergence theorem on a box

$F = (3x^2, 0, 0)$: $\nabla \cdot F = 6x$. Flux out of the unit cube $[0,1]^3$: $\int_0^1 \int_0^1 \int_0^1 6x dx dy dz = 3$. Direct check: the x=1 face carries $3(1)^2 \cdot (1 \cdot 1) = 3$ out, the x=0 face 0, the other faces are parallel to F – total 3 ✓. The divergence theorem is a bookkeeping identity, and the surface check makes it visible.

Curl and Stokes

$F = (-y, x, 0)$ (a rigid rotation): $\nabla \times F = (0, 0, 2)$ – twice the angular velocity. Circulation around the unit circle: $\oint$`F · dl = ∫(-sin`θ`, cos`θ`) · (-sin`θ`, cos`θ`)d`θ `= 2`π; Stokes gives $\int$`(∇×F) · dA = 2 · π · 1² = 2`π ✓. Curl measures local rotation; Stokes adds it up.

2 Practice

2.1 Compute $\nabla \cdot F$ for `F = (xy, yz, zx)` at the point (1, 2, 3). [2]

2.2 For $F = (x, y, 0)$: compute the flux out of the unit cube $[0,1]^3$ two ways (volume integral and surface check). [3]

2.3 Is $F = (y, 0, 0)$ rotation-free (zero curl)? Compute $\nabla \times F$. [2]

3 Exam-style questions

3.1 $F = (2x, -2y, 2z)$.

Compute $\nabla \cdot F$ and the flux out of the unit cube $[-1,1]^3$ by the volume integral; verify by computing the flux through each of the six faces directly. Then compute $\nabla \times F$ and interpret the field's structure (sources, sinks, rotation) in one sentence. [7]

3.2 A radial field `F = f̂ f(r)` with $f(r) = k/r^2$ (the inverse-square form).

Compute the flux through a sphere of radius R centered at the origin, and explain why the answer is independent of R. Then evaluate the flux through a cube of side 2R centered at the origin (same result – why?), and compute $\nabla \cdot F$ for $r \neq 0$. State what happens at the origin. [7]