6.7 The z-transform

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS z-transform z • unit circle

Objective

Compute z-transforms from the definition and tables, invert by partial fractions and long division, map poles to stability on the unit circle, and close the loop between the s- and z-planes.

1 Key knowledge and worked examples

■ The definition and the key pair

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

The workhorse pair: $a^n u[n] \leftrightarrow 1/(1-az^{-1})$, ROC `|z| > |a|`. Poles INSIDE the unit circle (given a causal ROC) mean BIBO stability – the discrete left-half-plane rule.

■ Worked example: the accumulator, three ways

`y[n] = 0.5y[n-1] + x[n]` $\rightarrow H(z) = 1/(1-0.5z^{-1})$, pole at $z = 0.5$ (inside the unit circle: stable ✓). Step response: $Y(z) = H(z) \cdot 1/(1-z^{-1})$; partial fractions give $y[n] = 2(1-0.5^{n+1})$ – at $n = 0$: `2(0.5) = 1` ✓ (the iteration's first value), $n \rightarrow \infty$ gives `2` ✓ (the DC gain). The three routes – iteration (6.6), transform, and steady-state formula – agree sample for sample.

■ The plane mapping

$s = j\omega$ maps to $z = e^{j\omega T}$ (the unit circle); $s = \sigma$ maps to radius $e^{\sigma T}$. So a discrete pole at radius `r` corresponds to a continuous decay `-ln(r)/T`: the z-plane is the s-plane wrapped around the origin – the sampling of EE06 unit 4, expressed in pole geometry.

2 Practice

2.1 Give the z-transform and ROC of $\delta[n]$, of `u[n]`, and of $(0.25)^n u[n]$. [3]

2.2 Invert $X(z) = 1/(1 - 2z^{-1})$ (causal): what is `x[n]`, and is it stable? [3]

2.3 A discrete pole sits at radius 0.368 with $T = 0.01s$. Give the equivalent continuous decay rate. [2]

3 Exam-style questions

3.1 $H(z) = (1 + z^{-1})/(1 - 0.5z^{-1})$ driven by a step.

Compute the impulse response `h[n]` (split the numerator: $H = 1/(1-0.5z^{-1}) + z^{-1}/(1-0.5z^{-1})$), the step response `y[n]` (partial fractions or one-shot division), and verify `y[0]` and `y[1]` by iterating the underlying difference equation `y[n] = 0.5y[n-1] + x[n] + x[n-1]`. State the DC gain from `H(1)` and check it against the step response's limit. [8]

3.2 An anti-averager: $H(z) = (1 - z^{-1})/2$ – `y[n] = (x[n] - x[n-1])/2`.

Compute the frequency response magnitude at $\omega = 0$ (z = 1) and $\omega = \pi$ (z = −1), verify both by direct substitution of constant and alternating inputs, and state the filter type. Then compute `|H|` at $\omega = \pi/2$ (z = j: $|1 - (-j)|$...*use*$e^{j\pi/2} = j$) and sketch/describe the magnitude curve from 0 to π. [7]