

6.6 Discrete-time systems

Name: _____ Class: _____ Date: _____

Total: 23 marks
KEY WORDS difference equation • characteristic equation • unit circle

Objective

Solve difference equations by iteration and by characteristic mode, compute impulse responses of discrete systems, and verify stability by the unit-circle root test.

1 Key knowledge and worked examples

■ Difference equations

The discrete counterpart of the ODE: $y[n] = \sum a_k y[n-k] + \sum b_m x[n-m]$. The **homogeneous** modes come from the **characteristic equation** $1 - a_1 z^{-1} - \dots = 0$: each root r gives r^n . Iterating from the initial conditions is the direct route and the check.

■ Worked example: the leaky accumulator

$y[n] = 0.5y[n-1] + x[n]$, $y[-1] = 0$, step input: 1, 1.5, 1.75, 1.875, ... converging to $1/(1-0.5) = 2$ (the DC gain formula $\sum h[n] = H(1)$ for this system = $1/(1-0.5)$). The mode 0.5^n decays – after 10 steps the remaining gap is $0.5^{10} \approx 0.1\%$. The machine check and the closed form $y[n] = 2(1 - 0.5^{n+1})$ agree sample for sample.

■ Discrete stability

BIBO stability $\Leftrightarrow$ every characteristic root satisfies $|r| < 1$ (inside the **unit circle**) – the discrete image of the left half-plane. A root on the circle is marginal (an oscillator or an integrator); outside, unbounded.

2 Practice

2.1 $y[n] = 0.8y[n-1] + x[n]$, $y[-1] = 0$, $x = u[n]$: iterate the first four values and state the limit. [3]

2.2 Which systems are stable: root 0.9; root -1.2 ; roots $0.5 \pm j0.5$? [3]

2.3 The impulse response $h[n] = (0.5)^n u[n]$: is the system stable? Why? [2]

3 Exam-style questions

3.1 $y[n] = -0.25y[n-2] + x[n]$ (a two-step recursion), $y[-1] = y[-2] = 0$, $x = u[n]$.

Iterate $y[0..6]$, solve the characteristic equation $r^2 = -0.25$ for the modes, and match the closed form $y[n] = A + B(0.5)^n \cos(n\pi/2 + \varphi)$ -type solution to your iterates (find the pattern's period). State the DC gain by the steady-state substitution $y = Y$ constant and verify against the iteration's trend. [8]

3.2 A moving-average system: $y[n] = (x[n] + x[n-1] + x[n-2])/3$.

Compute the impulse response $h[n]$, the response to the step, and the DC gain. Then compute the frequency response at $\omega = 0$ and at $\omega = \pi$ (alternating input $x[n] = (-1)^n$) and verify both by direct substitution of those inputs into the difference equation. [7]
