

6.2 LTI systems and convolution

Name: _____ Class: _____ Date: _____ **Total: 21 marks**

KEY WORDS convolution • impulse response

Objective

Compute convolutions graphically and analytically, use the width and endpoint properties as checks, and read the impulse response's meaning (memory, cause, stability) from the result.

1 Key knowledge and worked examples

■ Convolution is superposition made mechanical

$y(t) = \int x(\tau)h(t - \tau)d\tau$: flip h, slide, multiply, integrate. The checks that catch most errors:

- **Width** : supports of widths W_1, W_2 convolve to width $W_1 + W_2$.
- **Endpoints**: y starts when the two signals first touch and ends when they last overlap.
- **Area**: the area of $\mathbf{x}*\mathbf{h} = (\text{area of } \mathbf{x}) \times (\text{area of } \mathbf{h})$.

■ Worked example: step into exponential

$\mathbf{x} = \mathbf{u}(\mathbf{t})$, $h = e^{-t}u(t)$: $y(t) = \int_0^t e^{-(t-\tau)}d\tau = 1 - e^{-t}$ for $t \geq 0$. Endpoint checks: $y(0) = 0$ (no overlap before input starts) ✓; $y(\infty) = 1$ (DC gain: $H(0) = 1$) ✓. The same computation with $\mathbf{x} = \mathbf{rect}$ of width 1 gives the classic saturate-then-decay shape – saturating while the pulse is on, decaying after it ends – with the decay starting exactly at $t = 1$ (the width check).

■ Discrete convolution and the cascade property

$y[n] = \sum x[k]h[n - k]$ – same slide-multiply-SUM. The impulse response of a cascade is the convolution of the stages' impulse responses ($h = h_1 * h_2$), and parallel systems ADD their impulse responses. These two composition rules replace all circuit-style analysis for LTI systems.

2 Practice

2.1 h_1 and h_2 are both rectangles of width 2 starting at 0. Sketch/ describe $h_1 * h_2$ (shape, endpoints, peak value). [3]

2.2 Convolve $\mathbf{x}[n] = \{1, 2, 3\}$ (starting $n = 0$) with $\mathbf{h}[n] = \{1, 1\}$ ($n = 0, 1$). Give every output value and the support. [3]

2.3 A system's $\mathbf{h}(\mathbf{t}) = 0$ for $\mathbf{t} < 0$ and $h(t) = e^{-t}$ after. What does h tell you about causality and BIBO stability? [2]

3 Exam-style questions

3.1 $x(t) = e^{-2t}u(t)$ into $h(t) = e^{-3t}u(t)$.

Compute $\mathbf{y}(\mathbf{t}) = \mathbf{x}*\mathbf{h}$ analytically (keep the result as a difference of exponentials), verify the endpoints $y(0)$ and the decay tail, and verify the area property (areas $1/2 \times 1/3 = 1/6$ – check $\int ydt$). [6]

3.2 A two-stage LTI system: $h_1(t) = \delta(t) + \delta(t - 1)$ (a feedforward echo), $h_2(t) = u(t)$ (an integrator), with h_2 after h_1 .

Compute the overall impulse response $h = h_1 * h_2$, the step response (the system's response to $u(t)$), and predict the response to $x(t) = \delta(t) + \delta(t - 2)$. State whether the overall system is BIBO stable, with the integral test. [7]
