

7.4 Maxwell's equations

Name: _____ Class: _____ Date: _____

Total: 23 marks

KEY WORDS displacement current

Objective

Assemble the four Maxwell equations, explain displacement current through the charging capacitor, and use Faraday's law to compute induced EMF – every answer checked against continuity and the static limits.

1 Key knowledge and worked examples

■ The four equations

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \quad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t$$

Gauss for $\mathbf{E}$ (charge is the source of divergence), Gauss for $\mathbf{B}$ (no monopoles), **Faraday current** (changing $\mathbf{B}$ curls $\mathbf{E}$), Ampère–Maxwell (current PLUS **displacement** curls $\mathbf{B}$).

■ Worked example: the capacitor closes the loop

A capacitor (plates A) charging at $dQ/dt = 1A$: a conduction current of 1 A flows in the wires, zero between the plates. Ampère's loop around the wire must read 1 A whether the surface it spans cuts the wire or passes BETWEEN the plates. The displacement current supplies the difference: $E = Q/(\varepsilon_0 A)$, so $I_d = \varepsilon_0 A \cdot \partial E/\partial t = dQ/dt = 1A$ ✓ – exactly the missing ampere, with no free charge between the plates. Continuity is preserved: $\nabla \cdot (J + \varepsilon_0 \partial E/\partial t) = 0$ even while $\nabla \cdot J = -\partial \rho/\partial t$ at the plate.

■ Faraday: flux change induces EMF

$EMF = -d\Phi/dt$ (the minus sign is Lenz: the induced current opposes the change). Worked example: a 100-turn coil of area 10 cm^2 in a field rising linearly $0 \rightarrow 50 \text{ mT}$ in 20 ms: $d\Phi/dt$ per turn = $(0.05/0.02) \times 10^{-3} = 2.5 \text{ mWb/s}$, so $EMF = 100 \times 2.5 = 250 \text{ mV}$. No change means no EMF; faster change, proportionally more.

2 Practice

2.1 Name each Maxwell equation in words and state its static limit (what it reduces to when all time derivatives are zero). [4]

2.2 A parallel-plate capacitor ($A = 0.01 \text{ m}^2$) charges at 2 A: compute the displacement current between the plates. [2]

2.3 A single-loop coil of area 25 cm^2 sits in B falling from 80 mT to zero in 40 ms: compute the EMF. [3]

3 Exam-style questions

3.1 A circular parallel-plate capacitor (radius 5 cm, separation 1 mm) charges at a constant $I = 0.5 \text{ A}$.

Compute $E(t)$ between the plates (assume uniform), the displacement current density, and $B(r)$ between the plates at radius r (Ampère–Maxwell on a circular path): show $B(r) = \mu_0 I dr / (2\pi R^2)$ inside the plate radius R . Evaluate B at $r = R$ for this capacitor, and verify the magnitude against the wire’s field just outside (same current) – the two must match at the plate edge. [7]

3.2 A conducting rod of length 20 cm slides at 4 m/s on rails perpendicular to a uniform 0.8 T field (the rail generator).

Compute the motional EMF. The rails close through a $2\ \Omega$ resistor: find the current, the retarding force on the rod, and the mechanical power needed to keep it moving – then verify that power equals the resistor’s dissipation (energy conservation, the check that closes every such problem). State the direction of the induced current (Lenz). [7]

Solutions

2.1

- Gauss E: charge makes E diverge; static: Coulomb's law.
- Gauss B: no magnetic monopoles; static: closed B field lines.
- Faraday: changing B curls E; static: E is conservative (electrostatics).
- Ampère–Maxwell: current AND changing E curl B; static: Ampère's law of magnetostatics.

2.2

- $I_d = \varepsilon_0 A \partial E / \partial t = \partial Q / \partial t = 2A$ – equal to the conduction current, always.

2.3

- $EMF = A |dB/dt| = 25 \times 10^{-4} \times (0.08/0.04) = 5.0 \text{ mV}$.

3.1

- $E = Q/(\varepsilon_0 A)$ with $dQ/dt = 0.5 \rightarrow \partial E/\partial t = I/(\varepsilon_0 A)$.
- $J_d = \varepsilon_0 \partial E/\partial t = I/A = 0.5/(\pi \times 0.05^2) = 63.7 \text{ A/m}^2$.
- Ampère–Maxwell on radius r : $B \cdot 2\pi r = \mu_0 J_d \pi r^2$ (only displacement current threads the surface) $\rightarrow B = \mu_0 J_d r/2 = \mu_0 I r/(2A) = \mu_0 I r/(2\pi R^2) \checkmark$.
- At $r = R$: $B = \mu_0 I/(2\pi R) = 4\pi \times 10^{-7} \times 0.5/(2\pi \times 0.05) = 2\mu T$.
- The wire's field at 5 cm with the same 0.5 A: $\mu_0 I/(2\pi \times 0.05) = 2\mu T$ – identical at the plate edge $\checkmark$ (B is continuous across the boundary).

3.2

- Motional EMF: $Blv = 0.8 \times 0.20 \times 4 = 0.64V$.
- Current: $0.64/2 = 0.32 \text{ A}$.
- Retarding force: $F = BIl = 0.8 \times 0.32 \times 0.20 = 0.0512N$.
- Mechanical power: $P = Fv = 0.0512 \times 4 = 0.205W$.
- Resistor: $P = I^2 R = 0.32^2 \times 2 = 0.205W \checkmark$ – energy conserved exactly.
- Direction (Lenz): the induced current circulates so its field OPPOSES the increase of flux through the loop – clockwise as seen with B into the page and the rod moving right (a generator you must push against).