

7.2 Electrostatics

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS Gauss's law • potential • capacitance • dielectric

Objective

Compute fields by Gauss's law (sphere, cylinder, plane), potentials by integration, capacitance by Q/V , and apply dielectric boundary conditions – every Gauss answer checked at the boundary and in the far field.

1 Key knowledge and worked examples

■ Gauss's law chooses the surface

$$\oint \mathbf{E} \cdot d\mathbf{A} = Q_{enc}/\epsilon_0$$

Symmetry picks a surface where E is constant and normal: spheres for point/spherical charge, cylinders for line charge, pillboxes for sheets. The method fails when symmetry breaks – then integrate Coulomb directly.

■ Worked example: the uniformly charged sphere

Inside ($r < a$): $E = \rho r/(3\epsilon_0)$ (enclosed charge $\propto r^3$, area $\propto r^2$). Outside: $E = \rho a^3/(3\epsilon_0 r^2)$ – the sphere acts as a point charge. Boundary check: both give $\rho a/3\epsilon_0$ at $r = a$ ✓. Far field: $Q/(4\pi\epsilon_0 r^2)$ with $Q = 4\pi a^3 \rho/3$ ✓. Potential (zero at infinity): outside $V = Q/(4\pi\epsilon_0 r)$; inside $V = Q(3a^2 - r^2)/(8\pi\epsilon_0 a^3)$ – continuous at $r = a$ and maximal at the center (a charged sphere is a potential hill with a flat top rounded off).

■ Capacitance and dielectrics

$C = Q/V$ – geometry in one number: the isolated sphere $C = 4\pi\epsilon_0 a$; parallel plates $C = \epsilon A/d$. A **dielectric** with ϵ_r scales $D = \epsilon E$ and reduces the field for the same free charge: C multiplies by ϵ_r . **Boundary conditions:** E tangential continuous; D normal continuous (no free surface charge) – so at an air–dielectric interface the normal E drops by ϵ_r .

2 Practice

2.1 A line charge λ (C/m): using a coaxial cylindrical surface, derive $E(r)$ and state its distance dependence. [3]

2.2 An infinite sheet of charge σ (C/m²): E via a pillbox, and the direction. [3]

2.3 A parallel-plate capacitor has $A = 1\text{cm}^2$, $d = 0.1\text{mm}$, $\epsilon_r = 4.7$ (glass): compute C in pF. [3]

3 Exam-style questions

3.1 A coaxial cable: inner conductor radius $a = 1\text{mm}$, outer conductor inner radius $b = 5\text{mm}$, dielectric $\epsilon_r = 2.25$ (polyethylene) between.

Using Gauss's law in the dielectric, derive $E(r)$ for $a < r < b$ in terms of the charge per unit length λ . Then integrate to find the voltage between the conductors, and compute the capacitance per unit length $C' = \lambda/V$ (check the standard $C' = 2\pi\epsilon/\ln(b/a)$). Evaluate numerically in pF/m. [7]

3.2 A point charge $q = 10nC$ sits 5 cm above an infinite grounded conducting plane (the method of images).

State the image charge and its position. Compute the field's normal component at the point on the plane directly below the charge, and the total induced surface charge on the plane (integrate $\sigma = \varepsilon_0 E_n$ over the plane – the answer should equal the image charge). Compute the force on the real charge. [7]

Solutions

2.1

- Gauss cylinder: $E \cdot 2\pi rL = \lambda L/\epsilon_0$.
- $E = \lambda/(2\pi\epsilon_0 r)$ – falls as $1/r$.

2.2

- Pillbox: $2E \cdot A = \sigma A/\epsilon_0$.
- $E = \sigma/(2\epsilon_0)$, pointing away from the sheet on both sides.

2.3

- $C = \epsilon_r \epsilon_0 A/d = 4.7 \times 8.85p \times 1e - 4/1e - 4$.
- $= 4.7 \times 8.85pF = 41.6pF$.
- (Units: $A/d = 1e-4 \text{ m}^2 / 1e-4 \text{ m} = 1 \text{ m}$ – so the numeric is simply $4.7 \times 8.85pF/m \cdot m = 41.6 \text{ pF}$ ✓.)

3.1

- Gauss in the dielectric: $D \cdot 2\pi rL = \lambda L \rightarrow D = \lambda/(2\pi r)$, $E = \lambda/(2\pi\epsilon_0\epsilon_r r)$.
- Voltage: $V = \int_a^b E dr = \lambda/(2\pi\epsilon_0\epsilon_r) \cdot \ln(b/a)$.
- $C' = \lambda/V = 2\pi\epsilon_0\epsilon_r/\ln(b/a)$ ✓ the standard form.
- Numeric: $2\pi \times 8.85p \times 2.25/\ln(5) = 2\pi \times 19.9p/1.609 = 77.8pF/m$.

3.2

- Image: $-q = -10nC$ at 5 cm BELOW the plane (mirror position, opposite sign).
- At the point below the charge (distance 5 cm from each): the real charge's field there points down (toward the plane) with magnitude $q/(4\pi\epsilon_0 r^2) = 10n/(4\pi\epsilon_0 \times 0.05^2)$; the image's field points the same way (its field at that point also points down): total normal field $E_n = 2 \times 10n/(4\pi\epsilon_0 \times 0.0025) = 71.9kV/m$.
- Total induced charge: integrating $\sigma(r) = -qh/(2\pi(r^2 + h^2)^{3/2})$ over the plane gives exactly $-q = -10nC$ – equal to the image charge ✓ (that is the method's content).
- Force: attraction to the image at separation $2h = 0.1 \text{ m}$: $F = q^2/(4\pi\epsilon_0(2h)^2) = (10n)^2/(4\pi\epsilon_0 \times 0.01) = 89.9\mu N$, toward the plane.