

7.1 Vector field tools

Name: _____ Class: _____ Date: _____

Total: 21 marks

KEY WORDS flux

Objective

Choose coordinates by symmetry, compute divergence/curl/flux integrals, and verify every integral-theorem result by a direct surface or line check.

1 Key knowledge and worked examples

■ The symmetry method

- Spherical symmetry $\rightarrow$ spheres $dA = r^2 \sin\theta d\theta d\varphi$.
- Cylindrical $\rightarrow$ coaxial cylinders $dA = r d\varphi dl$.
- Planar $\rightarrow$ pillbox caps.
- No symmetry $\rightarrow$ direct integration.

■ Worked example: divergence theorem on a box

$F = (3x^2, 0, 0)$: $\nabla \cdot F = 6x$. Flux out of the unit cube $[0, 1]^3$: $\int_0^1 \int_0^1 \int_0^1 6x dx dy dz = 3$. Direct check: the $x=1$ face carries $3(1)^2 \cdot (1 \cdot 1) = 3$ out, the $x=0$ face 0, the other faces are parallel to F – total 3 ✓. The divergence theorem is a bookkeeping identity, and the surface check makes it visible.

■ Curl and Stokes

$F = (-y, x, 0)$ (a rigid rotation): $\nabla \times F = (0, 0, 2)$ – twice the angular velocity. Circulation around the unit circle: $\oint F \cdot d\mathbf{l} = \int (-\sin\theta, \cos\theta) \cdot (-\sin\theta, \cos\theta) d\theta = 2\pi$; Stokes gives $\int (\nabla \times F) \cdot d\mathbf{A} = 2 \cdot \pi \cdot 1^2 = 2\pi$ ✓. Curl measures local rotation; Stokes adds it up.

2 Practice

2.1 Compute $\nabla \cdot F$ for $F = (xy, yz, zx)$ at the point $(1, 2, 3)$.

[2]

2.2 For $F = (x, y, 0)$: compute the flux out of the unit cube $[0, 1]^3$ two ways (volume integral and surface check). [3]

2.3 Is $F = (y, 0, 0)$ rotation-free (zero curl)? Compute $\nabla \times F$. [2]

3 Exam-style questions

3.1 $F = (2x, -2y, 2z)$.

Compute $\nabla \cdot F$ and the flux out of the unit cube $[-1, 1]^3$ by the volume integral; verify by computing the flux through each of the six faces directly. Then compute $\nabla \times F$ and interpret the field's structure (sources, sinks, rotation) in one sentence. [7]

3.2 A radial field $\mathbf{F} = \hat{\mathbf{r}} f(r)$ with $f(r) = k/r^2$ (the inverse-square form).

Compute the flux through a sphere of radius R centered at the origin, and explain why the answer is independent of R . Then evaluate the flux through a cube of side $2R$ centered at the origin (same result – why?), and compute $\nabla \cdot \mathbf{F}$ for $r \neq 0$. State what happens at the origin. [7]

Solutions

2.1

- $\nabla \cdot F = y + z + x$ at $(1,2,3)$: $2 + 3 + 1 = 6$.

2.2

- $\nabla \cdot F = 1 + 1 + 0 = 2$; volume integral: $2 \times 1 = 2$.
- Surface check: $x=1$ face $F = (1, y, 0)$, outward normal $(+1, 0, 0)$, flux $1 \cdot 1 = 1$; $x=0$ face $F = (0, y, 0)$, flux 0 (normal $(-1, 0, 0)$, no x component); $y=1$ face similarly 1; $y=0$ face 0; z -faces 0 (F has no z). Total $1 + 1 = 2$ ✓ matches the volume integral.

2.3

- $\nabla \times F = (0, 0, \partial F_x / \partial y) = (0, 0, 1)$ – NOT rotation-free.

3.1

- $\nabla \cdot F = 2 - 2 + 2 = 2$.
- Volume: $2 \times 8 = 16$.
- Faces: $x=1$: $2 \cdot 4 = 8$ out; $x=-1$: $F = (-2, \dots)$ with outward normal $(-1, 0, 0) \rightarrow (+2) \cdot 4 = 8$ out; $y=\pm 1$: $\mp 2 \times 4$ with normal $(\mp)$ $\rightarrow -8 + (-8)??$ – carefully: $y=1$: $F_y = -2$, normal $(0, 1, 0) \rightarrow -2 \cdot 4 = -8$ (INWARD); $y=-1$: $F_y = +2$, normal $(0, -1, 0) \rightarrow -8$ (inward); $z=\pm 1$: $\pm 2 \cdot 4$ with matching normals $\rightarrow +8 + 8$. Total: $8 + 8 - 8 - 8 + 8 + 8 = 16$ ✓.
- $\nabla \times F = 0$ – irrotational.
- Structure: a saddle-like source/sink pattern (outflow along $\pm x$ and $\pm z$, inflow along $\pm y$) with no local rotation.

3.2

- Sphere flux: $f(r) \cdot 4\pi r^2 = (k/R^2) \cdot 4\pi R^2 = 4\pi k$.
- Independent of R because the field's $1/r^2$ fall exactly cancels the r^2 growth of the area.
- Cube: the divergence is zero everywhere except the origin, so any closed surface enclosing the origin carries the same total $-4\pi k$ (the flux through the cube equals the sphere's).
- $\nabla \cdot F = 0$ for $r \neq 0$ (compute in spherical: $(1/r^2) d(r^2 f) / dr = (1/r^2) d(k) / dr = 0$).
- At the origin the field is singular – the divergence is a delta function $4\pi k \delta^3(r)$, the point source that Gauss's law charges count.