

6.7 The z-transform

Name: _____ Class: _____ Date: _____

Total: 23 marks

KEY WORDS z-transform z • unit circle

Objective

Compute z-transforms from the definition and tables, invert by partial fractions and long division, map poles to stability on the unit circle, and close the loop between the s- and z-planes.

1 Key knowledge and worked examples

■ The definition and the key pair

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

The workhorse pair: $a^n u[n] \leftrightarrow 1/(1 - az^{-1})$, ROC $|z| > |a|$. Poles INSIDE the unit circle (given a causal ROC) mean BIBO stability – the discrete left-half-plane rule.

■ Worked example: the accumulator, three ways

$y[n] = 0.5y[n-1] + x[n] \rightarrow H(z) = 1/(1 - 0.5z^{-1})$, pole at $z = 0.5$ (inside the unit circle: stable ✓). Step response: $Y(z) = H(z) \cdot 1/(1 - z^{-1})$; partial fractions give $y[n] = 2(1 - 0.5^{n+1})$ – at $n = 0$: $2(0.5) = 1$ ✓ (the iteration's first value), $n \rightarrow \infty$ gives 2 ✓ (the DC gain). The three routes – iteration (6.6), transform, and steady-state formula – agree sample for sample.

■ The plane mapping

$s = j\omega$ maps to $z = e^{j\omega T}$ (the unit circle); $s = \sigma$ maps to radius $e^{\sigma T}$. So a discrete pole at radius r corresponds to a continuous decay $-\ln(r)/T$: the z-plane is the s-plane wrapped around the origin – the sampling of EE06 unit 4, expressed in pole geometry.

2 Practice

2.1 Give the z-transform and ROC of $\delta[n]$, of $u[n]$, and of $(0.25)^n u[n]$. [3]

2.2 Invert $X(z) = 1/(1 - 2z^{-1})$ (causal): what is $x[n]$, and is it stable? [3]

2.3 A discrete pole sits at radius 0.368 with $T = 0.01s$. Give the equivalent continuous decay rate. [2]

3 Exam-style questions

3.1 $H(z) = (1 + z^{-1})/(1 - 0.5z^{-1})$ driven by a step.

Compute the impulse response $h[n]$ (split the numerator: $H = 1/(1 - 0.5z^{-1}) + z^{-1}/(1 - 0.5z^{-1})$), the step response $y[n]$ (partial fractions or one-shot division), and verify $y[0]$ and $y[1]$ by iterating the underlying difference equation $y[n] = 0.5y[n-1] + x[n] + x[n-1]$. State the DC gain from $H(1)$ and check it against the step response's limit. [8]

3.2 An anti-averager: $H(z) = (1 - z^{-1})/2$ - $y[n] = (x[n] - x[n-1])/2$.

Compute the frequency response magnitude at $\omega = 0$ ($z = 1$) and $\omega = \pi$ ($z = -1$), verify both by direct substitution of constant and alternating inputs, and state the filter type. Then compute $|H|$ at $\omega = \pi/2$ ($z = j$: $|1 - (-j)| \dots use e^{j\pi/2} = j$) and sketch/describe the magnitude curve from 0 to π . [7]

Solutions

2.1

- $\delta[n] \leftrightarrow 1$ (all z).
- $u[n] \leftrightarrow 1/(1 - z^{-1})$, $|z| > 1$.
- $(0.25)^n \leftrightarrow 1/(1 - 0.25z^{-1})$, $|z| > 0.25$.

2.2

- $x[n] = 2^n u[n]$.
- Not stable: the pole 2 is outside the unit circle – $\sum |2^n|$ diverges.

2.3

- $\sigma = -\ln(0.368)/0.01 = -100 \cdot (-1.0) = \dots$ – compute: $\ln(0.368) \approx -1.0$ ($0.368 \approx e^{-1}$), so $\sigma \approx -1/0.01 = -100s^{-1}$.

3.1

- $h[n] = 0.5^n u[n] + 0.5^{n-1} u[n-1]$.
- Step: $Y = H(z) \cdot 1/(1 - z^{-1})$; partial fractions give $y[n] = 3 - 2 \cdot 0.5^{n+1} \dots$ – compute cleanly: $Y = (1+z^{-1})/((1-0.5z^{-1})(1-z^{-1})) = A/(1-0.5z^{-1}) + B/(1-z^{-1})$; cover-up at $z^{-1} = 2$: $A = (1+2)/(1-2) = -3$; at $z^{-1} = 1$: $B = (1+1)/(1-0.5) = 4$. So $y[n] = -3(0.5)^n + 4$ for $n \geq 0$.
- Check $y[0] = -3 + 4 = 1 \checkmark$; $y[1] = -1.5 + 4 = 2.5$.
- Iteration: $y[0] = 0.5(0) + 1 + 0 = 1 \checkmark$; $y[1] = 0.5(1) + 1 + 1 = 2.5 \checkmark$.
- DC gain: $H(1) = (1+1)/(1-0.5) = 4$ – the step response's limit $\checkmark$.

3.2

- $|H(z)| = |1 - z^{-1}|/2$: at $z = 1$ ($\omega = 0$): $|0|/2 = 0$.
- At $z = -1$ ($\omega = \pi$): $|1 - (-1)|/2 = 1$.
- Constant input: $y = (1-1)/2 = 0 \checkmark$ zero DC gain. Alternating: $y[n] = ((-1)^n - (-1)^{n-1})/2 = (-1)^n \checkmark$ gain 1.
- Filter type: high-pass differencer (null at DC, max at Nyquist).
- At $\omega = \pi/2$, $z = j$: $|1 - 1/j| = |1 + j| = \sqrt{2}$, so $|H| = \sqrt{2}/2 \approx 0.707$.
- Curve: 0 at $\omega = 0$, rising monotonically ($\sin(\omega/2)$ shape: $|H| = |\sin(\omega/2)|$) to 1 at π .