6.6 Discrete-time systems

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS difference equation • characteristic equation • unit circle

Objective

Solve difference equations by iteration and by characteristic mode, compute impulse responses of discrete systems, and verify stability by the unit-circle root test.

1 Key knowledge and worked examples

■ Difference equations

The discrete counterpart of the ODE: $y[n] = \sum a_k y[n-k] + \sum b_m x[n-m]$. The **homogeneous** modes come from the **characteristic equation** $1 - a_1 z^{-1} - \ldots = 0$: each root `r` gives r^n. Iterating from the initial conditions is the direct route and the check.

■ Worked example: the leaky accumulator

`y[n] = 0.5y[n-1] + x[n]`, `y[-1] = 0`, step input: `1, 1.5, 1.75, 1.875, …` converging to $1/(1-0.5) = 2$ (the DC gain formula $\sum h[n] = H(1)$ for this system $= 1/(1-0.5)$). The mode 0.5^n decays – after 10 steps the remaining gap is $0.5^{10} \approx 0.1\%$. The machine check and the closed form $y[n] = 2(1 - 0.5^{n+1})$ agree sample for sample.

■ Discrete stability

BIBO stability ⇔ every characteristic root satisfies `|r| < 1` (inside the **unit circle**) – the discrete image of the left half-plane. A root on the circle is marginal (an oscillator or an integrator); outside, unbounded.

2 Practice

2.1 `y[n] = 0.8y[n-1] + x[n]`, `y[-1] = 0`, `x = u[n]`: iterate the first four values and state the limit. [3]

2.2 Which systems are stable: root 0.9; root -1.2; roots $0.5 \pm j0.5$? [3]

2.3 The impulse response $h[n] = (0.5)^n u[n]$: is the system stable? Why? [2]

3 Exam-style questions

3.1 `y[n] = -0.25y[n-2] + x[n]` (a two-step recursion), `y[-1] = y[-2] = 0`, `x = u[n]`.

Iterate `y[0..6]`, solve the characteristic equation $r^2 = -0.25$ for the modes, and match the closed form $y[n] = A + B(0.5)^n cos(n\pi/2 + \varphi)$-type solution to your iterates (find the pattern's period). State the DC gain by the steady-state substitution `y = Y` constant and verify against the iteration's trend. [8]

3.2 A moving-average system: `y[n] = (x[n] + x[n-1] + x[n-2])/3`.

Compute the impulse response h[n], the response to the step, and the DC gain. Then compute the frequency response at $\omega = 0$ and at $\omega = \pi$ (alternating input $x[n] = (-1)^n$) and verify both by direct substitution of those inputs into the difference equation. [7]

Solutions

2.1

- `y[0] = 1, y[1] = 1.8, y[2] = 2.44, y[3] = 2.952`.
- Limit `1/(1-0.8) = 5`.

2.2

- `|0.9| < 1`: stable.
- `|-1.2| > 1`: unstable.
- $|0.5 \pm j0.5| = \sqrt{0.5} \approx 0.707 < 1$: stable.

2.3

- Stable: $\sum |0.5|^n = 2 < \infty$ (absolutely summable).

3.1

- Iterates (each step uses $y[n] = 1 - 0.25 \cdot y[n-2]$): `y[0] = 1, y[1] = 1` then `y[2] = 0.75, y[3] = 0.75`, then `y[4] = 0.8125, y[5] = 0.8125` and `y[6] = 0.7969`.
- Characteristic: $r^2 + 0.25 = 0 \rightarrow r = \pm j0.5$ – magnitude 0.5, angle $\pm 90°$: the modes oscillate with period 4 samples ($(0.5)^n$ envelope, $cos(n\pi/2)$ factor).
- The iterates repeat in PAIRS (y[0]=y[1], y[2]=y[3], …) ✓ consistent with the period-4 mode plus the constant.
- DC gain: $Y = -0.25Y + 1 \rightarrow 1.25Y = 1$, $Y = 0.8$; the iterates oscillate around and converge to 0.8 (0.7969 at n = 6) ✓ trend matches.

3.2

- $h[n] = (\delta[n] + \delta[n-1] + \delta[n-2])/3 \rightarrow$ `{1/3, 1/3, 1/3}` for n = 0,1,2.
- Step response: running sum `{1/3, 2/3, 1, 1, 1, …}` – settles at 1 after two steps.
- DC gain $H(1) = \sum h = 1$ ✓ (matches the step limit).
- At $\omega = 0$ (constant x = 1): $y = (1+1+1)/3 = 1$ ✓ gain 1. $y[n] = ((-1)^n + (-1)^{n-1} + (-1)^{n-2})/3$ equals $(-1)^n/3$ – gain 1/3, i.e. $|H(e^{j\pi})| = 1/3$: the averager attenuates alternating input ✓.