

6.5 Laplace transform and system functions

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS system function • poles and zeros • frequency response

Objective

Move between $\mathbf{h}(\mathbf{t})$, $\mathbf{H}(\mathbf{s})$ and $H(j\omega)$; read natural modes from poles; assess stability from the pole locations; and verify time- and frequency-domain answers against each other.

1 Key knowledge and worked examples

■ The system function

$\mathbf{H}(\mathbf{s}) = \mathbf{Y}(\mathbf{s})/\mathbf{X}(\mathbf{s})$, and $\mathbf{h}(\mathbf{t}) \leftrightarrow \mathbf{H}(\mathbf{s})$. For rational $\mathbf{H}(\mathbf{s}) = \text{poles/zeros} + \text{constant}$: each real pole $\mathbf{p}$ contributes a mode e^{pt} to the impulse response; complex poles contribute decaying sinusoids at the pole angle. The ROC must include the $j\omega$ axis for a stable system – equivalently, all poles in the open left half-plane.

■ Worked example: one pole, three views

$\mathbf{H}(\mathbf{s}) = 1/(\mathbf{s}+1)$:

- $h(t) = e^{-t}u(t)$ (pole at -1 is the exponent).
- Step response $1/(\mathbf{s}(\mathbf{s}+1)) = 1/\mathbf{s} - 1/(\mathbf{s}+1) \rightarrow 1 - e^{-t}$ – the unit 6.2 result, now by algebra.
- **Frequency response** $H(j\omega) = 1/(1 + j\omega)$: gain $1/\sqrt{2}$ at $\omega = 1$ (the pole magnitude), phase -45° there, -90° at ∞ .

One pole, three consistent views – the checks write themselves.

■ Composition in the s-domain

Cascade: $H = H_1 \cdot H_2$ (poles and zeros combine). Feedback $H = G/(1 + GH)$ – the denominator's roots are the closed-loop poles, and their left-half-plane membership is the stability verdict (the subject EE10 takes to design).

2 Practice

2.1 $\mathbf{H}(\mathbf{s}) = 10/(\mathbf{s}+5)$: give $\mathbf{h}(\mathbf{t})$, the DC gain, and the -3 dB frequency. [3]

2.2 A system has poles at $s = -2$ and $s = -1 \pm j10$. Write the form of its natural response (no coefficients). [3]

2.3 $\mathbf{H}(\mathbf{s}) = 1/((\mathbf{s}+1)(\mathbf{s}-2))$: is the causal system stable? Why (one sentence)? [2]

3 Exam-style questions

3.1 An LTI system: $H(s) = s/(s^2 + 2s + 101)$.

Factor the poles, write the general impulse response (with unknown coefficients), and solve them from $h(0^+) = 0$ and the initial slope $h'(0^+) = 1$ (the transform's high-frequency behaviour). Then compute $|H(j\omega)|$ at $\omega = 10$ and at $\omega = 0$, and state the type of filter this is. [8]

3.2 A unity-feedback system: open loop $G(s) = K/((s + 1)(s + 2))$.

Compute the closed-loop $H(s)$, the poles as a function of K , and the stability condition on K (Routh or direct root inspection). Then evaluate the pole locations at $K = 2$ and verify the DC closed-loop gain is $H(0) = 2/(2+2) = 1/2$. [7]

Solutions

2.1

- $h(t) = 10e^{-5t}u(t)$.
- DC gain $H(0) = 10/5 = 2$.
- -3 dB at $\omega = 5$ (the pole magnitude).

2.2

- $Ae^{-2t} + e^{-t}(B\cos 10t + C\sin 10t)$.

2.3

- Not stable: the pole at $s = +2$ (right half-plane) gives a growing e^{+2t} mode – any bounded input with that component makes the output diverge.

3.1

- Poles: $s = -1 \pm j10$.
- $h(t) = e^{-t}(A\cos 10t + B\sin 10t)$ for $t \geq 0$.
- $h(0^+) = A = 0$.
- Initial-slope condition: $H(s) \approx 1/s$ for large s (leading numerator matches leading denominator degree), which inverts to an impulse at 0 – more usefully, $h'(0^+)$: with $A = 0$, $h = Be^{-t}\sin 10t$, $h'(0) = 10B = 1 \rightarrow B = 0.1$. So $h(t) = 0.1e^{-t}\sin(10t)u(t)$.
- $|H(j10)| = |j10/((-100 + 2j10 + 101))| = 10/|1 + j20| = 10/20.02 \approx 0.5$.
- $|H(j0)| = 0/101 = 0$.
- Filter type: band-pass (zero at DC, resonant peak near $\omega = 10$, falling at high frequency).

3.2

- Closed loop: $H = G/(1 + G) = K/(s^2 + 3s + 2 + K)$.
- Poles: $s = [-3 \pm \sqrt{9 - 4(2 + K)}]/2$.
- Stability: both poles in LHP iff $9 - 4(2 + K) < \infty$ (always) AND real parts negative iff $2 + K > 0$ – i.e. $K > -2$. For $K > 1/4$ the poles are complex (still stable); Routh confirms: second order needs all coefficients positive $\rightarrow K > -2$.
- At $K = 2$: $s = (-3 \pm \sqrt{9 - 16})/2 = -1.5 \pm j1.32$.
- DC gain: $H(0) = 2/(2 + 2) = 0.5$ ✓ – the tracking fraction of the feedback law.