6.5 Laplace transform and system functions

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS system function • poles and zeros • frequency response

Objective

Move between `h(t)`, `H(s)` and $H(j\omega)$; read natural modes from poles; assess stability from the pole locations; and verify time- and frequency-domain answers against each other.

1 Key knowledge and worked examples

■ The system function

`H(s) = Y(s)/X(s)`, and `h(t)` $\leftrightarrow$ `H(s)`. For rational `H(s)` = poles/zeros + constant: each real pole `p` contributes a mode e^{pt} to the impulse response; complex poles contribute decaying sinusoids at the pole angle. The ROC must include the $j\omega$ axis for a stable system – equivalently, all poles in the open left half-plane.

■ Worked example: one pole, three views

`H(s) = 1/(s+1)`:

- $h(t) = e^{-t}u(t)$ (pole at -1 is the exponent).
- Step response `1/(s(s+1)) = 1/s - 1/(s+1)` $\rightarrow 1 - e^{-t}$ – the unit 6.2 result, now by algebra.
- **Frequency response** $H(j\omega) = 1/(1 + j\omega)$: gain $1/\sqrt{2}$ at $\omega = 1$ (the pole magnitude), phase $-45°$ there, $-90°$ at ∞.

One pole, three consistent views – the checks write themselves.

■ Composition in the s-domain

Cascade: $H = H_1 \cdot H_2$ (poles and zeros combine). Feedback $H = G/(1 + GH)$ – the denominator's roots are the closed-loop poles, and their left-half-plane membership is the stability verdict (the subject EE10 takes to design).

2 Practice

2.1 `H(s) = 10/(s+5)`: give `h(t)`, the DC gain, and the -3 dB frequency. [3]

2.2 A system has poles at $s = -2$ and $s = -1 \pm j10$. Write the form of its natural response (no coefficients). [3]

2.3 `H(s) = 1/((s+1)(s-2))`: is the causal system stable? Why (one sentence)? [2]

3 Exam-style questions

3.1 An LTI system: $H(s) = s/(s^2 + 2s + 101)$.

Factor the poles, write the general impulse response (with unknown coefficients), and solve them from $h(0^+) = 0$ and the initial slope $h'(0^+) = 1$ (the transform's high-frequency behaviour). Then compute $|H(j\omega)|$ at $\omega = 10$ and at $\omega = 0$, and state the type of filter this is. [8]

3.2 A unity-feedback system: open loop $G(s) = K/((s+1)(s+2))$.

Compute the closed-loop `H(s)`, the poles as a function of K, and the stability condition on K (Routh or direct root inspection). Then evaluate the pole locations at $K = 2$ and verify the DC closed-loop gain is `H(0) = 2/(2+2) = 1/2`. [7]

Solutions

2.1

- $h(t) = 10e^{-5t}u(t)$.
- DC gain `H(0) = 10/5 = 2`.
- -3 dB at $\omega = 5$ (the pole magnitude).

2.2

- $Ae^{-2t} + e^{-t}(Bcos10t + Csin10t)$.

2.3

- Not stable: the pole at $s = +2$ (right half-plane) gives a growing e^{+2t} mode – any bounded input with that component makes the output diverge.

3.1

- Poles: $s = -1 \pm j10$.
- $h(t) = e^{-t}(Acos10t + Bsin10t)$ for $t \geq 0$.
- $h(0^+) = A = 0$.
- Initial-slope condition: $H(s) \approx 1/s$ for large s (leading numerator matches leading denominator degree), which inverts to an impulse at 0 – more usefully, $h'(0^+)$: with $A = 0$, $h = Be^{-t}sin10t$, $h'(0) = 10B = 1 \rightarrow B = 0.1$. So $h(t) = 0.1e^{-t}sin(10t)u(t)$.
- $|H(j10)| = |j10/((-100 + 2j10 + 101))| = 10/|1 + j20| = 10/20.02 \approx 0.5$.
- `|H(j0)| = 0/101 = 0`.
- Filter type: band-pass (zero at DC, resonant peak near $\omega = 10$, falling at high frequency).

3.2

- Closed loop: $H = G/(1 + G) = K/(s^2 + 3s + 2 + K)$.
- Poles: $s = [-3 \pm \sqrt{(9 - 4(2 + K))}]/2$.
- Stability: both poles in LHP iff `9 - 4(2+K) < ∞` (always) AND real parts negative iff `2 + K > 0` – i.e. `K > -2`. For `K > 1/4` the poles are complex (still stable); Routh confirms: second order needs all coefficients positive $\rightarrow$ `K > -2`.
- At $K = 2$: $s = (-3 \pm \sqrt{9 - 16})/2 = -1.5 \pm j1.32$.
- DC gain: `H(0) = 2/(2 + 2) = 0.5` ✓ – the tracking fraction of the feedback law.