

6.4 Sampling

Name: _____ Class: _____ Date: _____ **Total: 21 marks**

KEY WORDS Nyquist criterion • aliasing • anti-alias filter

Objective

Apply the Nyquist criterion, compute aliased frequencies, design the anti-alias filter, and reconstruct from samples – every aliasing answer verified sample by sample.

1 Key knowledge and worked examples

■ Sampling replicates the spectrum

Sampling at f_s multiplies the signal by an impulse train: the spectrum replicates at $0, \pm f_s, \pm 2f_s, \dots$. If the signal is band-limited to $f_{max} < f_s/2$, the replicas do not overlap and the original is recoverable – the **Nyquist criterion** $f_s > 2f_{max}$.

■ Worked example: the 4 kHz tone at 6 kHz

$f = 4kHz > f_s/2 = 3kHz$ – aliased. The apparent frequency is $|f - k \cdot f_s|$ brought into $[0, f_s/2]$: $|4 - 6| = 2$ kHz. Sample check: $\cos(2\pi \cdot 4000 \cdot n/6000) = \cos(2\pi \cdot (2/3)n)$ and $\cos(2\pi \cdot 2000 \cdot n/6000) = \cos(2\pi \cdot (1/3)n)$; since $\cos(2\pi \cdot (2/3)n) = \cos(2\pi n - 2\pi \cdot (1/3)n) = \cos(2\pi \cdot (1/3)n)$ (cos is even, $2\pi n$ is a whole number of turns) – the sequences are identical sample by sample ✓.

■ Anti-alias BEFORE the sampler; reconstruct after

The **anti-alias filter** must band-limit the signal before sampling – afterwards the aliases are already folded in and no filter can separate them. Reconstruction: an ideal brick-wall at $f_s/2$ (sinc interpolation in time); practical DACs approximate it (zero-order hold plus smoothing).

2 Practice

2.1 A signal is band-limited to 18 kHz. State the minimum sampling rate (theoretical) and a practical rate with guard band. [2]

2.2 A 5 kHz tone is sampled at 8 kHz. Compute the aliased frequency and verify with three actual samples of both sequences. [3]

2.3 Why can an anti-alias filter never help AFTER sampling? One sentence. [2]

3 Exam-style questions

3.1 An audio signal contains components up to 20 kHz plus an unwanted 38 kHz interference. The system samples at 44.1 kHz.

Compute where the 38 kHz component lands (aliased frequency), and design the anti-alias filter: state the passband edge, the stopband edge that kills the interference before it folds, and the minimum filter order for a first-order cascade to give ≥ 40 dB attenuation at the stopband edge (each first-order section gives -20 dB/decade). [7]

3.2 A sample-and-hold measurement system samples a 90 Hz vibration at $f_s = 200Hz$.

Compute the apparent frequency of the vibration (it exceeds $f_s/2 = 100...$ verify: $90 < 100$ – no aliasing). Now a harmonic at 130 Hz appears: compute its alias, and verify by evaluating three samples of the 130 Hz tone against your claimed alias. State the Nyquist-safe band for this system and what sampling rate would raise the safe band to 200 Hz. [7]

Solutions

2.1

- Theoretical minimum: $2 \times 18k = 36kHz$.
- Practical: e.g. 44.1 kHz or 48 kHz (guard band for the reconstruction filter).

2.2

- $f_s/2 = 4kHz < 5kHz$: aliased to $|5 - 8| = 3 \text{ kHz}$.
- Samples: $\cos(2\pi \cdot 5n/8)$ at $n = 0,1,2 = 1$, $\cos(5\pi/4) = -0.707$, $\cos(5\pi/2) = 0$; $\cos(2\pi \cdot 3n/8) = 1$, $\cos(3\pi/4) = -0.707$, $\cos(3\pi/2) = 0$ – identical ✓.

2.3

- Once sampled, the aliases are folded INTO the band as legitimate low frequencies – indistinguishable from real content, so no later filter can remove them.

3.1

- $38kHz > f_s/2 = 22.05kHz$: alias at $44.1 - 38 = 6.1 \text{ kHz}$.
- Anti-alias passband edge: 20 kHz (the wanted content).
- The 38 kHz interference must be attenuated BEFORE it folds to 6.1 kHz – stopband must start at 22.05 kHz (anything above $f_s/2$ folds).
- Filter spans $20 \rightarrow 22.05 \text{ kHz}$ (a decade factor of $22.05/20 = 1.10$).
- -40 dB needs two first-order sections IF the stopband is a decade above the corner – but the ratio is only 1.10, so first-order sections give far less than 40 dB per decade here; the design must specify a sharp (higher-order elliptic/Bessel-style) filter or oversampling. The honest answer: at this narrow transition, first-order cascades are inadequate – the real fix is oversampling (e.g. 176.4 kHz) plus digital filtering.

3.2

- $90 \text{ Hz} < 100 \text{ Hz}$: NO aliasing – the vibration appears correctly at 90 Hz.
- $130 \text{ Hz} > 100 \text{ Hz}$: alias at $200 - 130 = 70 \text{ Hz}$.
- Samples: $\cos(2\pi \cdot 130n/200) = \cos(1.3\pi n)$ at $n = 0,1,2 = 1$, $\cos(1.3\pi) = -0.588$, $\cos(2.6\pi) = -0.309$; $\cos(2\pi \cdot 70n/200) = \cos(0.7\pi n) = 1$, $\cos(0.7\pi) = -0.588$, $\cos(1.4\pi) = -0.309$ – identical ✓ (the even-cos identity: $\cos(1.3\pi n) = \cos(2\pi n - 0.7\pi n) = \cos(0.7\pi n)$ for integer n).
- Nyquist-safe band: 0–100 Hz. To make 0–200 Hz safe: $f_s > 400Hz$.