

6.3 Fourier series and transforms

Name: _____ Class: _____ Date: _____

Total: 21 marks

KEY WORDS Fourier transform • duality

Objective

Compute Fourier transforms from the definition and properties, apply the convolution-to-multiplication rule, and reason with bandwidth and duality.

1 Key knowledge and worked examples

■ The pair and the working properties

$$X(\omega) = \int x(t)e^{-j\omega t} dt \quad x(t) = \frac{1}{2\pi} \int X(\omega)e^{j\omega t} d\omega$$

The properties that do the work: **time shift** $x(t-t_0) \leftrightarrow X(\omega)e^{-j\omega t_0}$; **frequency shift** (modulation) $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$; **duality** (swap the roles); **convolution** $x*h \leftrightarrow X \cdot H$ – multiplication in frequency; and the symmetric pair $x \cdot y \leftrightarrow (1/2\pi)X*Y$.

■ Worked example: the rectangle and its sinc

A unit-height rectangle of width τ centred at 0 transforms to $X(\omega) = \tau \text{sinc}(\omega\tau/2)$. Checks: at $\omega = 0$ the transform equals the pulse's area τ ✓ ($\text{sinc}(0) = 1$); the first null sits at $\omega\tau/2 = \pi$, i.e. $f = 1/\tau$ – the narrower the pulse, the wider the spectrum. By DUALITY, a sinc-in-time transforms back to a rectangle in frequency – the ideal low-pass filter's impulse response, whose infinite extent is the price of a brick-wall frequency response.

■ Convolution in time = multiplication in frequency

The LTI input-output law $y = x*h$ becomes $Y(\omega) = X(\omega)H(\omega)$: filter by multiplying spectra. **Worked example:** $H(\omega) = 1/(1 + j\omega)$ (the $e^{-t}u(t)$ system): a cosine $\cos(\omega_0 t)$ input passes as $|H(\omega_0)|\cos(\omega_0 t + \angle H(\omega_0))$. At $\omega_0 = 1$: $|H| = 1/\sqrt{2}$ and $\angle H = -45^\circ$ – the -3 dB, -45° point, exactly as EE05's frequency response found. Two routes (time-domain convolution, frequency-domain multiplication) – one answer.

2 Practice

2.1 Using the time-shift property: if $x(t) \leftrightarrow X(\omega)$, give the transform of $x(t - 3)$ and of $x(t + 3)$. [2]

2.2 A signal $x(t) = \cos(2\pi \cdot 1000 \cdot t)$ is multiplied by $\cos(2\pi \cdot 10^4 \cdot t)$. Describe the spectrum of the product (which frequencies appear). [3]

2.3 State the duality pair for the rectangle $\rightarrow$ sinc result: what is $F\tau \text{sinc}(Wt/2)$ -shaped time signal's spectrum? [2]

3 Exam-style questions

3.1 Compute the Fourier transform of $x(t) = e^{-at}u(t)$ ($a > 0$) from the definition, give $|X(\omega)|$ and $\angle X(\omega)$, and state the half-power frequency. Then compute $Fte^{-at}u(t)$ using the property $t \cdot x(t) \leftrightarrow j \cdot dX/d\omega$, and verify both magnitudes at $\omega = 0$ against the areas of the two time signals. [7]

3.2 An ideal low-pass filter $H(\omega) = 1$ for $|\omega| \leq W$, 0 otherwise, is driven by $x(t) = \text{sinc}^2(Wt/\pi)$ – whose spectrum is a triangle peaking at $2\pi/T$... carefully: $F\text{sinc}^2(at) = (\pi/a) \cdot \text{tri}(\omega/2a)$ (triangle in frequency).

Take $a = W/\pi$ so the triangle's base spans exactly $2W$. Compute the output spectrum $Y(\omega) = X \cdot H$ (triangle truncated at $|\omega| = W$ – i.e. half the triangle), and explain why the output must be even and real. State, without computing it, what happens to the output if W is doubled (filter and input together). [7]

Solutions

2.1

- $x(t - 3) \leftrightarrow X(\omega)e^{-j3\omega}$.
- $x(t + 3) \leftrightarrow X(\omega)e^{+j3\omega}$ (advance = opposite sign).

2.2

- Product of cosines = $\frac{1}{2}\cos$ of sum and difference: components at $10^4 \pm 10^3$ Hz = 9 kHz and 11 kHz.
- Spectrum: two impulses at ± 9 kHz and ± 11 kHz (each cos is a doublet; the multiplication convolves them into the four lines).

2.3

- A sinc in time $\leftrightarrow$ rectangle in frequency: $F\tau \text{sinc}(Wt/2\pi)\dots$ – the symmetric statement of the pair.

3.1

- $X(\omega) = \int_0^\infty e^{-at}e^{-j\omega t}dt = 1/(a + j\omega)$.
- $|X| = 1/\sqrt{a^2 + \omega^2}$; $\angle X = -\text{atan}(\omega/a)$.
- Half-power at $\omega = a$: $|X| = 1/\sqrt{2}$ (the pole frequency, as in the handout).
- $Fte^{-at}u(t) = j \cdot d/d\omega[1/(a + j\omega)] = j \cdot (-1)(a + j\omega)^{-2} \cdot j = 1/(a + j\omega)^2$.
- Areas: $\int e^{-at} = 1/a = X(0) \checkmark$; $\int te^{-at} = 1/a^2 = |X_t(0)| = 1/a^2 \checkmark$ (both DC values equal the time areas).

3.2

- $Y(\omega) = X(\omega) \cdot H(\omega)$: the triangle $\mathbf{X}$ times the rectangle $\mathbf{H}$ = the triangle truncated to $|\omega| \leq W$ – the positive half of the base retained, the outer slopes removed.
- Even and real: $\mathbf{x(t)}$ is even and real, so $X(\omega)$ is even and real; $\mathbf{H}$ is even and real; their product is even and real – and a real even spectrum inverts to a real even time signal $\checkmark$.
- Doubling W : the filter passes the whole triangle (base $2W \leq 2W_{\text{new}}$) AND the input triangle doubles its base – the output spectrum becomes the full doubled triangle; the time signal narrows (inverse scaling).