UNIVERSITY ELECTRICAL ENGINEERING • EXERCISE SHEET

6.2 LTI systems and convolution

Name: _____ Class: _____ Date: _____ **Total: 21 marks**

KEY WORDS convolution • impulse response

Objective

Compute convolutions graphically and analytically, use the width and endpoint properties as checks, and read the impulse response's meaning (memory, cause, stability) from the result.

1 Key knowledge and worked examples

■ Convolution is superposition made mechanical

$y(t) = \int x(\tau)h(t-\tau)d\tau$: flip h, slide, multiply, integrate. The checks that catch most errors:

- **Width** : supports of widths W_1, W_2 convolve to width $W_1 + W_2$.
- **Endpoints**: `y` starts when the two signals first touch and ends when they last overlap.
- **Area**: the area of `x*h` = (area of x) × (area of h).

■ Worked example: step into exponential

`x = u(t)`, $h = e^{-t}u(t)$: $y(t) = \int_0^t e^{-(t-\tau)}d\tau = 1-e^{-t}$ for $t \geq 0$. Endpoint checks: `y(0) = 0` (no overlap before input starts) ✓; `y(∞) = 1` (DC gain: `H(0) = 1`) ✓. The same computation with `x = rect` of width 1 gives the classic saturate-then-decay shape – saturating while the pulse is on, decaying after it ends – with the decay starting exactly at $t = 1$ (the width check).

■ Discrete convolution and the cascade property

$y[n] = \sum x[k]h[n-k]$ – same slide-multiply-SUM. The impulse response of a cascade is the convolution of the stages' impulse responses ($h = h_1 * h_2$), and parallel systems ADD their impulse responses. These two composition rules replace all circuit-style analysis for LTI systems.

2 Practice

2.1 h_1 and h_2 are both rectangles of width 2 starting at 0. Sketch/ describe $h_1 * h_2$ (shape, endpoints, peak value). [3]

2.2 Convolve `x[n] = {1, 2, 3}` (starting n = 0) with `h[n] = {1, 1}` (n = 0, 1). Give every output value and the support. [3]

2.3 A system's `h(t) = 0` for `t < 0` and $h(t) = e^{-t}$ after. What does h tell you about causality and BIBO stability? [2]

3 Exam-style questions

3.1 $x(t) = e^{-2t}u(t)$ into $h(t) = e^{-3t}u(t)$.

Compute `y(t) = x*h` analytically (keep the result as a difference of exponentials), verify the endpoints `y(0)` and the decay tail, and verify the area property (areas $1/2 \times 1/3 = 1/6$ – check $\int y dt$). [6]

3.2 A two-stage LTI system: $h_1(t) = \delta(t) + \delta(t-1)$ (a feedforward echo), $h_2(t) = u(t)$ (an integrator), with h_2 after h_1.

Compute the overall impulse response $h = h_1 * h_2$, the step response (the system's response to `u(t)`), and predict the response to $x(t) = \delta(t) + \delta(t-2)$. State whether the overall system is BIBO stable, with the integral test. [7]

Solutions

2.1

- Triangle: starts at 0, peaks at t = 2, ends at t = 4 (width 2+2 = 4 ✓).
- Peak value: overlap of the two rectangles at t = 2 = 2 (integral of a height-1 rectangle of overlap length 2).

2.2

- `y[0] = 1`; $y[1] = 1 \cdot 1 + 2 \cdot 1 = 3$; `y[2] = 2 + 3 = 5`; `y[3] = 3`.
- Support n = 0..3 (lengths 3 and 2 → 4 samples ✓). (– width check, no marks)

2.3

- Causal ✓ (h is zero before 0).
- BIBO stable ✓ ($\int e\hat{}\{-t\} = 1 < \infty$).

3.1

- $y(t) = \int_0^t e^{-2\tau} e^{-3(t-\tau)}\, d\tau = e^{-3t}(e^t - 1) = e^{-2t} - e^{-3t}$ for $t \geq 0$.
- `y(0) = 1 - 1 = 0` ✓ (no overlap before 0).
- Tail: e^{-2t} dominates (decays) ✓ – the slower exponent wins, as it must.
- Area: $\int_0^\infty (e^{-2t} - e^{-3t})dt = 1/2 - 1/3 = 1/6 =$ `(1/2)(1/3)` ✓ the product of the input areas.

3.2

- $h = h_1 * h_2 = u(t) + u(t-1)$ (convolving a shifted delta with h_2 shifts h_2).
- Step response: $y = u * h = (u * u)(t) + (u * u)(t-1) = t + (t-1)u(t-1)$ (a ramp starting at 0 plus one starting at 1).
- $x = \delta(t) + \delta(t-2) \to h(t) + h(t-2) = u(t) + u(t-1) + u(t-2) + u(t-3)$ (a staircase: +1 each at 0, 1, 2, 3).
- BIBO: $\int |h| = \int (u + u(t-1))dt$ diverges – unstable (the integrator makes any constant input diverge) ✓.