6.1 Signal representations and system properties

Name: _____ Class: _____ Date: _____ **Total: 22 marks**

KEY WORDS signal • system • linearity • time invariance • causality • BIBO stability BIBO

Objective

Classify signals and systems by the four properties, apply the shift-then-system versus system-then-shift test, and verify LTI behaviour from the definition.

1 Key knowledge and worked examples

■ **Four properties, tested from the definition**

- **Linearity** : $a \cdot x_1 + b \cdot x_2 \to a \cdot y_1 + b \cdot y_2$ – test with superposition.
- **Time invariance** : shift input by $t_0 \Rightarrow$ output shifts by exactly t_0 – compare "shift then system" with "system then shift".
- **Causality** : no output before input – check `y(t)` for `t < 0` on an input starting at 0.
- **BIBO stability** BIBO : bounded in, bounded out.

■ **Worked example: the time-scaling trap**

Is $y(t) = x(2t)$ time-invariant? Input `u(t)`: $y = u(2t) = u(t)$ (same step). Shift the input: $x(t-1) = u(t-1)$ gives $y = u(2t-1) = u(t-½)$. But shifting the original OUTPUT gives $y(t-1) = u(2(t-1)) = u(2t-2) = u(t-1)$. Half shift versus full shift – different: **time-varying** ✓ (time scaling rescales shifts). Systems that are memoryless and of the form `y = f(x(t))` with fixed f ARE time-invariant; it is the argument scaling that breaks the property.

■ **Stability from the impulse response**

For LTI systems, BIBO $\Leftrightarrow \int |h(t)|dt < \infty$ (the impulse response is absolutely integrable). Worked checks: $h = e^{-t}u(t)$ – integrable, stable ✓; `h = u(t)` (an integrator) – the integral diverges, unstable (a ramp input diverges) ✓; `h = sin(t)/t` – NOT absolutely integrable, yet bounded outputs: the subtlety that motivates the strict

definition. Causality from `h`: `h(t) = 0` for `t < 0`.

2 Practice

2.1 Classify each as linear/nonlinear: $y = 3x + 1$; $y = t \cdot x(t)$; $y = x^2$. [3]

2.2 Is $y(t) = x(t) + 5$ time-invariant? Causal? [2]

2.3 Which impulse responses are BIBO-stable: $e^{-2t}u(t)$, `u(t)`, $e^{-t}cos(t)u(t)$? [3]

3 Exam-style questions

3.1 Test each system on all four properties (linearity, time invariance, causality, BIBO stability), with one-line evidence each:

(a) $y(t) = x(t-2)$; (b) $y(t) = t \cdot x(t)$; (c) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$; (d) $y[n] = x[n]^2$. [8]

3.2 A system is claimed LTI. Two measurement pairs: `x_1 = δ(t) → y_1 = e^{-t}u(t)`; `x_2 = δ(t-1) → y_2 = e^{-(t-1)}u(t-1)`.

Verify the second response is the LTI-predicted one. Then predict the response to $x_3 = 2\delta(t) + 3\delta(t-1)$ and check the claimed system's BIBO stability from its impulse response. State whether $y(t) = x(t) \cdot h(t)$ would be a valid output rule for this system (and why not). [6]

Solutions

2.1

- `3x + 1`: NOT linear (the constant fails zero-in-zero-out).
- $t \cdot x(t)$: linear (scales with input; t is a coefficient, not a nonlinearity).
- x^2: nonlinear.

2.2

- Time-invariant ✓ (no argument scaling; the +5 shifts with the system).
- Causal ✓ (output at t depends only on x(t)).

2.3

- $e^{-2t}u(t)$: stable ($\int = 1/2$).
- `u(t)`: unstable (integral diverges – integrator).
- $e^{-t}cos(t)u(t)$: stable (decays faster than 1/t; absolutely integrable).

3.1

- (a) Delay: linear ✓, time-invariant ✓, causal ✓, stable ✓ (memoryless bounded passthrough).
- (b) $t \cdot x(t)$: linear ✓; time-VARYING (the gain t changes with absolute time – test: shifting input does not shift the gain) ; causal ✓; NOT BIBO (x = 1 gives y = t, unbounded).
- (c) Integrator: linear ✓, time-invariant ✓, causal ✓, NOT BIBO (step input gives a ramp).
- (d) $x[n]^2$: nonlinear (superposition fails); time-invariant ✓; causal ✓; bounded input gives bounded output ✓ (squaring preserves boundedness) – stable but nonlinear, so the impulse-response test does not apply.

3.2

- LTI prediction: $h(t-1) = e^{-(t-1)}u(t-1)$ – exactly the measured y_2 ✓.
- $x_3 = 2\delta(t) + 3\delta(t-1) \rightarrow y_3 = 2e^{-t}u(t) + 3e^{-(t-1)}u(t-1)$ (superposition of scaled, shifted impulse responses).
- BIBO: $\int |h| = \int_0^\infty e^{-t}dt = 1 < \infty$ – stable ✓.
- $y = x \cdot h$ is NOT the rule for LTI systems (that would be pointwise multiplication, which is nonlinear in general); the correct law is convolution `y = x * h`.