UNIVERSITY ELECTRICAL ENGINEERING • EXERCISE SHEET

4.6 Laplace circuit analysis

Name: _____ Class: _____ Date: _____ **Total: 22 marks**

KEY WORDS Laplace transform • pole

Objective

Transform circuits with initial conditions into the s-domain, solve by all the DC methods, and read the response – poles, transients and steady state – from the algebra, verifying against time-domain endpoints.

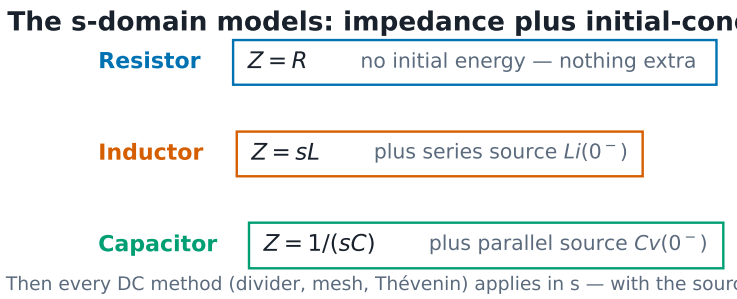
1 Key knowledge and worked examples

■ The s-domain element models

The **Laplace transform** turns each element into an impedance plus initial-condition sources:

- Resistor: `Z = R` (no initial energy).
- Inductor with $i(0^-)$: `Z = sL` in series with a voltage source $L \cdot i(0^-)$ (or parallel current source $i(0^-)/s$).
- Capacitor with $v(0^-)$: $Z = 1/(sC)$ in parallel with a current source $C \cdot v(0^-)$ (or series voltage source $v(0^-)/s$).

Then dividers, node and mesh all apply in s. Poles of the response = transient exponents; the input's poles = steady state.



The s-domain models: impedance plus initial-condition source.

■ Worked example: the RC step, with initial charge

$R = 10k\Omega$, $C = 100\mu F$ ($\tau = 1$ s), step 9 V, $v_C(0) = 1V$.

- s-domain KVL: $9/s = I \cdot R + v_C(0)/s + I/(sC)$.
- $I = (9-1)/(s(R + 1/(sC))) = 8/(10^4 s + 10^4) = 8/(10^4(s+1))$.
- $V_C = v_C(0)/s + I/(sC) = 1/s + 8/(s(s+1)) = 9/s - 8/(s+1)$.
- $v_C(t) = 9 - 8e^{-t}$ V – the wave-1 answer, now algebra.

Checks: $v_C(0) = 1$ ✓; $v_C(\infty) = 9$ ✓; the single pole at -1 gives the 1-second time constant ✓. The identity $1/s + 8/(s(s+1)) = 9/s - 8/(s+1)$ verifies at any s (e.g. $s = 1$: $1 + 4 = 5 = 9 - 4$ ✓).

■ What the poles say

A response $V(s)$ with poles at $s = 0$ (constant), $s = -a$ (decay e^{-at}), and $\pm j\omega$ (oscillation) reads its whole time shape before inversion. Initial conditions scale the residues; they never move the poles – the poles belong to the circuit, the residues to its history.

2 Practice

2.1 Transform each: a 4 Ω resistor; a 0.5 H inductor (no initial current); that inductor with `i(0) = 2 A` (impedance + series source value). [3]

2.2 A 2 F capacitor with `v(0) = 3 V`: give the impedance and the parallel current source value. [2]

2.3 `F(s) = 10/s - 10/(s+2)`: give `f(t)` and its two endpoint values. [3]

3 Exam-style questions

3.1 A series RL circuit ($R = 4\Omega$, $L = 0.2H$) with $i(0) = 1.5$ A is driven at $t = 0$ by a 24 V step.

Transform to the s-domain, solve for `I(s)`, and invert to `i(t)`. Verify `i(0)` and `i(∞)`, and give the time constant from the pole. Then recompute the steady state directly from the circuit (inductor as short) as an independent check. [7]

3.2 A capacitor $C = 0.5F$ charged to `v(0) = 10 V` discharges through $R = 2\Omega$ (no source).

Using the s-domain model (impedance + initial-condition source), solve for the capacitor voltage $V_C(s)$ and invert. Verify `v(0)` and `v(∞)`, give the pole, and state the energy dissipated in the resistor over all time (check against $\tfrac{1}{2}CV^2$). [7]

Solutions

2.1

- $Z_R = 4\Omega$.
- $Z_L = 0.5s\Omega$.
- With $i(0) = 2$ A: $Z_L = 0.5s$ in series with a voltage source $L \cdot i(0) = 0.5 \times 2 = 1$ V.

2.2

- $Z_C = 1/(2s)$; parallel current source $C \cdot v(0) = 2 \times 3 = 6$ A $\cdot$ s (appearing as 6 in the s-domain source).

2.3

- $f(t) = 10 - 10e^{-2t}$.
- $f(0) = 0$; $f(\infty) = 10$.

3.1

- s-domain KVL: $24/s + L \cdot i(0) = I(s)(R + sL) \rightarrow 24/s + 0.3 = I(s)(4 + 0.2s)$.
- $I(s) = (24 + 0.3s)/(s(4 + 0.2s)) = (120 + 1.5s)/(s(s + 20))$.
- Partial fractions: $I(s) = 6/s - 4.5/(s + 20)$.
- $i(t) = 6 - 4.5e^{-20t}$ A.
- $i(0) = 6 - 4.5 = 1.5$ ✓; $i(\infty) = 6$ ✓.
- Pole at -20: $\tau = 1/20 = 0.05$ s ($= L/R = 0.2/4$ ✓).
- Direct steady state: inductor shorts at DC, $i = 24/4 = 6A$ ✓.

3.2

- s-domain loop (initial voltage as the series source): $10/s = I(s)(R + 1/(sC))$, so $I(s) = (10/s)/(2 + 2/s) = 10/(2s + 2) = 5/(s + 1)$.
- Capacitor voltage (current leaves the + terminal, discharging): $V_C(s) = 10/s - I(s) \cdot (1/(sC)) = 10/s - (5/(s+1)) \cdot (2/s) = 10/s - 10/(s(s+1)) = 10/(s+1)$, so $v_C(t) = 10e^{-t}$ V.
- $v(0) = 10$ ✓; $v(\infty) = 0$ ✓; pole at -1, $\tau = RC = 1$ s ✓.
- Energy: $\int_0^\infty v^2/R dt = \int_0^\infty 100e^{-2t}/2 dt = 25J$; check $\frac{1}{2}CV^2 = 0.5 \times 0.5 \times 100 = 25J$ ✓ – all stored energy dissipates.