# 4.3 Second-order RLC circuits

Name: _____ Class: _____ Date: _____ **Total: 22 marks**

**KEY WORDS** natural frequency • damping • underdamped • Q factor • overshoot • settling time

## Objective

Classify second-order responses by the damping ratio, compute step-response specifications, and verify every solution against its initial and final values.

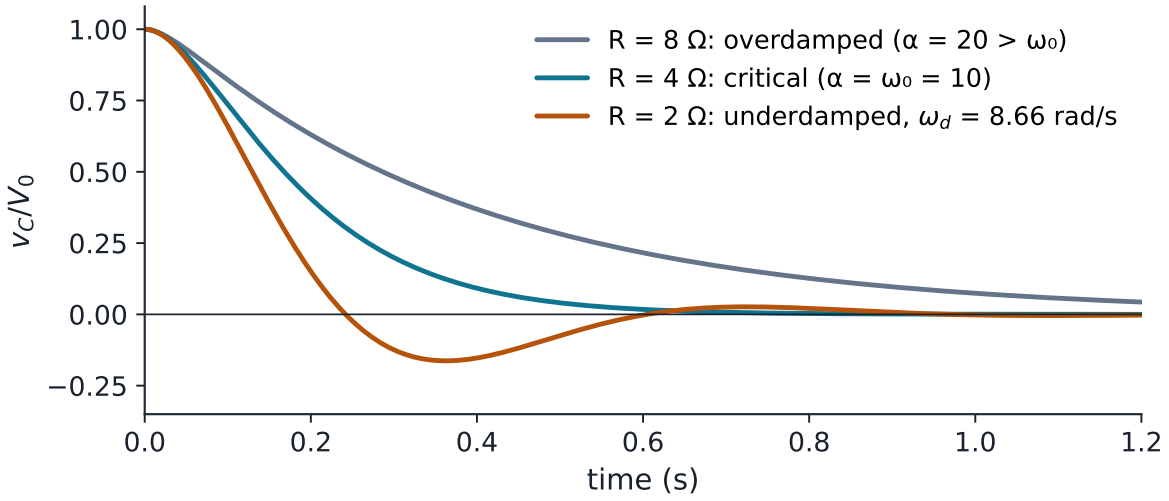
## 1 Key knowledge and worked examples

### ■ Two numbers decide everything

The series RLC natural response is governed by the **natural frequency** $\omega_0 = 1/\sqrt{LC}$ and the **damping** $\alpha = R/(2L)$:

- **Overdamped** ($\alpha > \omega_0$): $Ae^{s_1 t} + Be^{s_2 t}$, two real decays.
- **Critically damped** ($\alpha = \omega_0$): $(A+Bt)e^{-\alpha t}$ – fastest without oscillation.
- **Underdamped** ($\alpha < \omega_0$): $e^{-\alpha t}(A cos\omega_d t + B sin\omega_d t)$, ringing at $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.

The **Q factor** $Q = \omega_0/(2\alpha)$ measures the ring persistence.




*One circuit, three resistors: overdamped, critical, underdamped.*

■ **Worked example: one circuit, two regimes**

$L = 0.2H$, $C = 0.05F$: $\omega_0 = 1/\sqrt{0.01} = 10rad/s$.

- $R = 4\Omega$: $\alpha = 4/0.4 = 10 = \omega_0$ – critically damped.
- $R = 2\Omega$: $\alpha = 5 < 10$ – underdamped, $\omega_d = \sqrt{100 - 25} = 8.66rad/s$, $Q = 10/10 = 1$.

The regime test is one comparison – but the check that earns marks is the endpoint check: at $t = 0$ the response must equal the initial condition, and `di/dt(0)` must match what the circuit allows (for an inductor current, zero slope at switch-on when driven from a voltage step through R).

■ **Step-response specifications**

For the underdamped step response: **overshoot** $M_p = e^{-\pi\alpha/\omega_d}$ (as a fraction); **settling time** (2%) $t_s \approx 4/\alpha$; rise time grows with damping. **Worked example**: with $\alpha = 5, \omega_d = 8.66$: $M_p = e^{-5\pi/8.66} = e^{-1.814} = 0.163$ (16.3%), $t_s \approx 0.8s$. Check $M_p$ bounds: $0 <$ M_p $< 1$ for any $\alpha > 0$ ✓; and $cos\omega_d t_p = ...$ the peak time $t_p = \pi/\omega_d = 0.363s$, inside the settling window – consistent ordering.

## 2 Practice

**2.1** $R = 60\Omega$, $L = 1H$, $C = 1/400F$: compute $\omega_0$, $\alpha$, name the regime. [3]

**2.2** Compute $\omega_d$ for the circuit of 2.1 if it is underdamped (or state the regime result instead). [2]

**2.3** An underdamped system has $\alpha = 2, \omega_0 = 10$. Compute Q, $M_p$ and the 2% settling time. [3]

## 3 Exam-style questions

**3.1** A series RLC with $R = 5\Omega$, $L = 0.5H$, $C = 0.02F$ starts relaxed and is driven at $t = 0$ by a 20 V step.

Compute $\omega_0$, $\alpha$, the regime, and the complete capacitor-voltage response $v_C(t)$. Verify against $v_C(0)$ and $dv_C/dt(0)$ (zero, for a relaxed capacitor) and the final value. [8]

**3.2** A control designer needs the second-order step response to settle (2%) within 0.5 s with overshoot below 10%.

Derive the required $\alpha$ from the settling spec and the minimum $\alpha/\omega_d$ from the overshoot spec ($M_p = e^{-\pi\alpha/\omega_d}$), then choose $\alpha$ and the constraint on $\omega_d$ (or $\omega_0$) for a system with $\omega_0 = 20 rad/s$ available. Verify your design computes $M_p$ and $t_s$ on target. [6]

## Solutions

**2.1**

- $\omega_0 = 1/\sqrt{1 \times 1/400} = 20 rad/s$.
- $\alpha = 60/(2 \times 1) = 30$.
- $\alpha > \omega_0$ ($30 > 20$): overdamped.

**2.2**

- Not applicable – overdamped has no oscillation frequency; the roots are real: $s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -30 \pm \sqrt{500} = -30 \pm 22.36$, i.e. $-7.64$ and $-52.36$.

**2.3**

- $Q = 10/4 = 2.5$.
- $\omega_d = \sqrt{100 - 4} = 9.80$; $M_p = e^{-2\pi/9.80} = e^{-0.641} = 0.527$ (53%).
- $t_s \approx 4/2 = 2s$.

**3.1**

- $\omega_0 = 1/\sqrt{0.5 \times 0.02} = 10 rad/s$; $\alpha = 5/(2 \times 0.5) = 5$.
- $\alpha < \omega_0$: underdamped; $\omega_d = \sqrt{100 - 25} = 8.66 rad/s$.
- Form: $v_C = 20 + e^{-5t}(Acos8.66t + Bsin8.66t)$.
- $v_C(0) = 0$: $20 + A = 0 \rightarrow A = -20$.
- $dv_C/dt(0) = 0$: derivative at 0 is $-5A + 8.66B = 100 + 8.66B = 0 \rightarrow B = -11.55$.
- $v_C(t) = 20 - e^{-5t}(20cos8.66t + 11.55sin8.66t)$ V.
- Checks: $v_C(0) = 20 - 20 = 0$ ✓; final value 20 V ✓ (transient decays); slope at 0: $-5(-20) + 8.66(-11.55) = 100 - 100 = 0$ ✓.

**3.2**

- Settling: $4/\alpha \leq 0.5 \rightarrow \alpha \geq 8$.
- Overshoot: $e^{-\pi\alpha/\omega_d} \leq 0.1 \rightarrow \pi\alpha/\omega_d \geq ln10 = 2.303 \rightarrow \omega_d \leq \pi\alpha/2.303 = 1.364\alpha$.
- With $\alpha = 8$: $\omega_d \leq 10.9$; and $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{400 - 64} = 18.3$ at $\omega_0 = 20$ – check direction: larger $\omega_d$ relative to $\alpha$ means MORE overshoot; $18.3 > 10.9$ fails the overshoot spec, so $\omega_0$ must satisfy $\sqrt{\omega_0^2 - 64} \leq 10.9 \rightarrow \omega_0^2 \leq 64 + 10.9^2 = 182.8 \rightarrow \omega_0 \leq 13.5$ rad/s.
- Design: $\alpha = 8$, $\omega_0 = 13$ (say): $\omega_d = \sqrt{169 - 64} = 10.25$; $M_p = e^{-8\pi/10.25} = e^{-2.45} = 0.086$ (8.6% $\leq$ 10% ✓); $t_s = 4/8 = 0.5s$ ✓ – both specs met.