2.8 Vector calculus

Name: _____ Class: _____ Date: _____ **Total: 21 marks**

KEY WORDS gradient • divergence • curl • divergence theorem • Stokes' theorem

Objective

Compute gradient, divergence and curl in Cartesian coordinates, use $E = -\nabla V$, and verify field results by a small-step line integral.

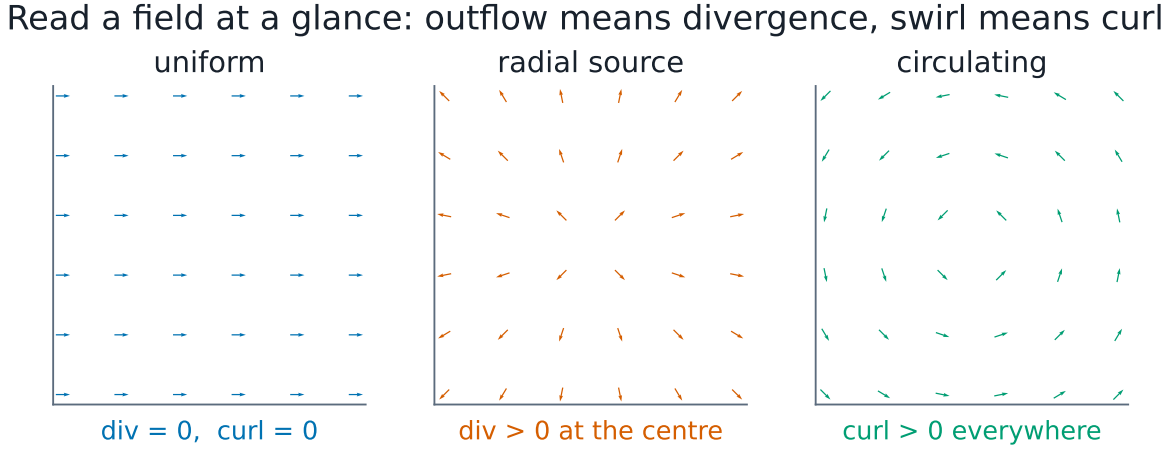
1 Key knowledge and worked examples

■ Three operators on fields

A vector field assigns a vector to each point; a scalar field a number. Three operators connect them:

- **Gradient** ∇V (scalar → vector): points up the steepest slope; the electric field is $E = -\nabla V$.
- **Divergence** $\nabla \cdot F$ (vector → scalar): net outflow per unit volume; for electric fields, the charge density.
- **Curl** $\nabla \times F$ (vector → vector): circulation per unit area; zero for electrostatics, the mechanism of induction where it is not.

For `F = (P, Q, R)` in Cartesian coordinates: $\nabla \cdot F = \partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$, and $\nabla V = (\partial V/\partial x, \partial V/\partial y, \partial V/\partial z)$.



■ Worked example: field from potential

$V(x, y) = 3x^2 y$ volts.

- $\nabla V = (6xy, 3x^2)$; at `(1, 2)`: $\nabla V = (12, 3)$, so $E = -(12, 3)$ V/m.
- Check by a small step: move $\delta = (0.01, 0)$ from `(1, 2)`; `V` changes by $\approx \partial V/\partial x \cdot \delta x = 12 \times 0.01 = 0.12$ V, and $-E \cdot \delta = 12 \times 0.01 = 0.12$ V ✓. The field-times-step predicts the potential drop – that IS the gradient.

■ Reading the two theorems

The **divergence theorem** converts a volume integral of $\nabla \cdot F$ into a closed-surface flux; **Stokes' theorem** converts a surface integral of $\nabla \times F$ into a loop circulation. Maxwell's equations are these two statements with physics poured in – Gauss's law is the divergence theorem; Faraday's law is Stokes.

2 Practice

2.1 For $V = 5x^2$: compute ∇V and `E`, and evaluate both at $x = 2$. [3]

2.2 For $F = (3x, 2y, 0)$: compute $\nabla \cdot F$ at `(1, 1, 0)` and say whether the point is a source or a sink. [3]

2.3 State the curl of any electrostatic field, and the physical law that changes this where magnetic fields vary. [2]

3 Exam-style questions

3.1 A potential is `V(x, y) = 10x − 5y` volts.

Compute `E`, state its magnitude and direction, and verify by computing the change in `V` along the step `(0.2, −0.1)` from the origin two ways: directly from `V`, and from $-E\cdot\delta$. [6]

3.2 The field $F = (2xy, x^2, 0)$ is claimed to be conservative (curl-free).

Compute $\nabla \times F$ to verify the claim. Then compute the line integral of `F` along the straight path from `(0,0)` to `(1,1)` parameterised as `(t, t)`, and confirm the same value along the path (t, t^2) – path independence being the signature of a conservative field. [7]

Solutions

2.1

- $\nabla V = (10x, 0)$; $E = -(10x, 0)$.
- At $x = 2$: $E = (-20, 0)$ V/m.

2.2

- $\nabla \cdot F = 3 + 2 + 0 = 5$.
- Positive divergence at every point: a source (net outflow).

2.3

- $\nabla \times E = 0$ for electrostatics.
- Faraday's law: a time-varying magnetic field makes $\nabla \times E = -\partial B/\partial t \neq 0$.

3.1

- $E = -\nabla V = -(10, -5) = (-10, 5)$ V/m.
- Magnitude $\sqrt{100 + 25} = 11.18 V/m$, direction `(-0.894, 0.447)`.
- Direct: `V(0.2, -0.1) - V(0,0) = 10(0.2) - 5(-0.1) = 2 + 0.5 = 2.5 V`.
- Field route: $-E \cdot \delta = -(-10 \times 0.2 + 5 \times (-0.1)) = -(-2 - 0.5) = 2.5V$ ✓ – wait, sign convention: $\Delta V = -E \cdot \delta = -(-10)(0.2) - (5)(-0.1) = 2 + 0.5 = 2.5V$ ✓.

3.2

- $\nabla \times F$ (z-component) $= \partial(x^2)/\partial x - \partial(2xy)/\partial y = 2x - 2x = 0$; the x, y components are zero by inspection (F has no z-dependence and $R = 0$).
- Path 1, `(t, t)`: $F = (2t \cdot t, t^2) = (2t^2, t^2)$; $dr = (1, 1)dt$; $\int F \cdot dr = \int_0^1 (2t^2 + t^2)dt = 3/3 = 1$.
- Path 2, (t, t^2): $F = (2t \cdot t^2, t^2) = (2t^3, t^2)$; $dr = (1, 2t)dt$; $\int F \cdot dr = \int_0^1 (2t^3 + 2t^3)dt = 4 \times 1/4 = 1$.
- Both paths give 1 ✓ – path independence, the signature of a conservative (curl-free) field.