

2.7 Probability and random variables

Name: _____ Class: _____ Date: _____

Total: 23 marks

KEY WORDS random variable • probability distribution • expectation
 • variance • standard deviation

Objective

Work with expectations, variance and standard deviation; apply the Gaussian noise model; and use the averaging rule $\sigma/\sqrt{n}$ with its instrument rationale.

1 Key knowledge and worked examples

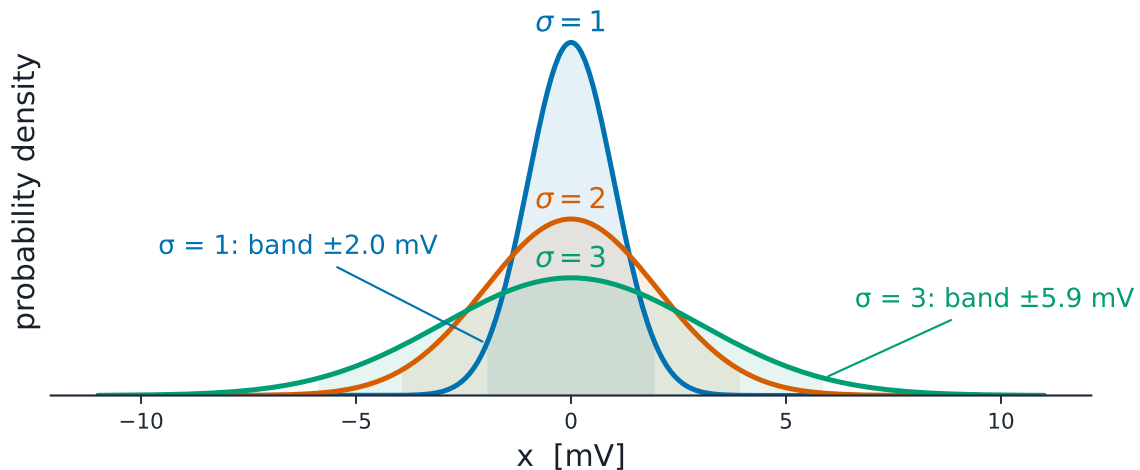
■ Distribution, mean, spread

A **random variable** X is described by its **probability distribution**. Two numbers summarise it: the **expectation** $E[X]$ (long-run average) and the **variance** $Var(X) = E[X^2] - \mu^2$, whose square root σ – the **standard deviation** – is the spread in the variable's own units.

$$X \sim N(\mu, \sigma^2) : f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

The Gaussian models thermal noise. Useful bands: $\pm 1\sigma \approx 68\%$, $\pm 2\sigma \approx 95\%$ (more exactly $\pm 1.96\sigma$), $\pm 3\sigma \approx 99.7\%$.

Three Gaussians, one area: each shaded band is the 95% interval $\pm 1.96 \sigma$



■ Worked example: reading a noise spec

A preamp's output noise is $N(0, (2\text{mV})^2)$: $E[N] = 0$, $\sigma = 2\text{mV}$.

- The 95% band is $\pm 1.96 \times 2 = \pm 3.9\text{mV}$ – a scope trace will sit inside this about 19 samples out of 20.
- Independent noise samples average: 16 averages give $\sigma/\sqrt{16} = 0.5\text{mV}$. Averaging is the cheapest noise filter there is, and $\sqrt{n}$ is why.

■ Variance adds; check with a die

For a fair die: $E[X] = 3.5$, $E[X^2] = (1 + 4 + 9 + 16 + 25 + 36)/6 = 15.17$, so $Var = 15.17 - 3.5^2 = 2.92$ and $\sigma = 1.71$. For the SUM of two dice, means add (7) and variances add (5.83), so $\sigma = 2.42$ – NOT 3.42: standard deviations do not add, variances do.

2 Practice

2.1 For $X \sim N(5, 4)$: state μ , σ , and the approximate 95% band.

[3]

2.2 A die's $\text{Var}(X) = 2.92$. Give the variance and standard deviation of the sum of 9 independent rolls, and the standard deviation of their AVERAGE. [4]

2.3 Noise with $\sigma = 6\text{mV}$ is averaged 36 times. Give the averaged noise's standard deviation and the factor of improvement. [3]

3 Exam-style questions

3.1 A sensor's readings carry noise $N(0, (3\text{mV})^2)$ about a true 10 mV signal.

For a single reading, give the probability the reading is within ± 3 mV of the true value, and within ± 6 mV. Then state how many independent averages bring the noise standard deviation to 0.5 mV, and the new $\pm 2\sigma$ band. [7]

3.2 Two independent noise sources $\sigma_1 = 2mV$ and $\sigma_2 = 3mV$ add at a node.

Compute the combined standard deviation, and explain why it is not 5 mV. Then compute the RMS of the combined noise and compare with $\sqrt{E[N^2]}$ to show the two coincide for zero-mean noise. [6]

Solutions

2.1

- $\mu = 5$, $\sigma = \sqrt{4} = 2$.
- 95% band: $5 \pm 1.96 \times 2 = 5 \pm 3.92$, i.e. about (1.08, 8.92).

2.2

- Sum of 9: variance $9 \times 2.92 = 26.3$.
- Sum's $\sigma = \sqrt{26.3} = 5.13$.
- Average's $\sigma = 5.13/9 = 0.57$ ($= \sigma/\sqrt{9} = 1.71/3$).

2.3

- $\sigma_{avg} = 6/\sqrt{36} = 1mV$.
- Improvement factor: $\sqrt{36} = 6$ (a 6 $\times$ reduction).

3.1

- Within $\pm 3 mV = \pm 1\sigma$: about 68%.
- Within $\pm 6 mV = \pm 2\sigma$: about 95%.
- Need $3/\sqrt{n} = 0.5 \rightarrow \sqrt{n} = 6 \rightarrow n = 36$ averages.
- New $\sigma = 0.5mV$.
- New $\pm 2\sigma$ band: $10 \pm 1.0mV$.
- Note the band narrowed from $\pm 6 mV$ to $\pm 1 mV$ – averaging bought a 6:1 improvement for 36 samples ($\sqrt{n}$ law).

3.2

- Variances add: $Var = 2^2 + 3^2 = 13(mV)^2$, so $\sigma = \sqrt{13} = 3.61mV$.
- Not 5 mV because standard deviations add only for perfectly correlated noise; independent sources add in quadrature (variances).
- RMS for zero-mean noise: $\sqrt{E[N^2]} = \sqrt{Var + 0^2} = \sqrt{13} = 3.61mV$.
- The two coincide because the mean is zero: $RMS = \sigma$ exactly for zero-mean noise.