

2.6 Fourier series

Name: _____ Class: _____ Date: _____

Total: 21 marks

KEY WORDS Fourier series • fundamental • harmonic

Objective

Compute Fourier coefficients for periodic signals, reconstruct signals from partial sums, and explain – with the numbers – why sharper edges need more harmonics.

1 Key knowledge and worked examples

■ Every periodic signal is a sum of sinusoids

For period T and **fundamental** frequency $\omega_0 = 2\pi/T$, the **Fourier series** is

$$f(t) = a_0 + \sum_{n \geq 1} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Each coefficient is an average of f against the matching sinusoid (orthogonality: distinct sinusoids average to zero against each other over a period). a_0 is just the mean value of f .

■ Worked example: the square wave

Odd symmetry kills every a_n ; the sine coefficients come out $b_n = 4/(n\pi)$ for odd n (even n gives zero):

$$f(t) = \frac{4}{\pi} \left(\sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \dots \right)$$

The check that convinces: at $t = T/4$ the true value is 1, and the series there is $4/\pi(1 - 1/3 + 1/5 - \dots) = 4/\pi \times \pi/4 = 1$ – the Gregory series converges to exactly the square wave's value. Each **harmonic** amplitude falls as $1/n$: the series converges slowly AT the edge (Gibbs ripple) and quickly elsewhere.

■ Symmetry is free work

- Odd f (like the square wave): all $a_n = 0$.
- Even f : all $b_n = 0$.
- Half-wave symmetry $f(t + T/2) = -f(t)$: only odd n survive.

Spot the symmetry before integrating – it deletes half the coefficients before any arithmetic begins.

2 Practice

2.1 State a_0 (the mean) for: a sine wave, a square wave ± 1 with equal halves, and a pulse train that is 1 for 25% of the period and 0 otherwise. [3]

2.2 The square wave has $b_3 = 4/(3\pi)$. Compute the first three harmonic amplitudes numerically (2 dp) and their ratio pattern. [3]

2.3 Name which coefficients vanish for an even function, and give one example signal. [2]

3 Exam-style questions

3.1 A pulse train is $f(t) = 3$ for $0 < t < T/4$ and 0 for $T/4 < t < T$.

Compute a_0 (the mean), and a_1 in terms of ω_0 (integral $\int 3\cos(\omega_0 t)dt = (3/\omega_0)\sin(\omega_0 t)$); give a_1 numerically for $T = 2ms$. State the units of a_0 relative to f . [7]

3.2 A filtered square wave keeps only its first two odd harmonics: $\frac{4}{\pi}\sin\omega_0 t + \frac{4}{3\pi}\sin 3\omega_0 t$.

Compute this partial sum at $\omega_0 t = \pi/2$ (the waveform's peak) and at $\omega_0 t = \pi/4$, and compare with the ideal square wave's values 1 and 1. Then explain, using the $1/n$ amplitude law, why a channel that passes only $n \leq 3$ cannot reproduce sharp edges. [6]

Solutions

2.1

- Sine wave: $a_0 = 0$.
- Symmetric square: $a_0 = 0$.
- 25% duty pulse train: $a_0 = 0.25 \times 1 = 0.25$.

2.2

- $b_1 = 4/\pi = 1.27$.
- $b_3 = 4/(3\pi) = 0.42$.
- $b_5 = 4/(5\pi) = 0.25$ – each odd harmonic is $1/n$ of the first (1, 1/3, 1/5).

2.3

- All b_n (the sine coefficients) vanish.
- Example: a symmetric triangular wave, or a full-wave rectified cosine.

3.1

- $a_0 = (1/T)(3 \times T/4) = 3/4$.
- $a_1 = (2/T) \int_0^{T/4} 3\cos(\omega_0 t) dt = (6/T)(1/\omega_0)[\sin\omega_0 t]_0^{T/4}$.
- $= (6/(T\omega_0))\sin(\pi/2) = 6/(T\omega_0) = 6/(2\pi) = 3/\pi = 0.955$.
- With $T = 2ms$: $\omega_0 = 2\pi/T = 1000\pi rad/s$, and a_1 is the dimensionless coefficient 0.955 (same units as f : 0.955 of the 3-unit amplitude).
- Symmetry note: this pulse train has neither pure odd nor even symmetry about 0, so sine terms also exist; only a_1 was asked.
- Check magnitude: $a_0 + a_1 = 0.75 + 0.955 = 1.705 < 3$ ✓ (partials cannot overshoot the peak by much).

3.2

- At $\omega_0 t = \pi/2$: $\sin(\pi/2) = 1$, $\sin(3\pi/2) = -1$, so the sum is $4/\pi - 4/(3\pi) = (4/\pi)(1 - 1/3) = 0.849$.
- At $\omega_0 t = \pi/4$: $\sin(\pi/4) = 0.707$, $\sin(3\pi/4) = 0.707$, sum $(4/\pi)(0.707)(1 + 1/3) = 1.200$.
- Ideal: 1 at both points – the partial sum misses by 15% low and 20% high, and the overshoot is the Gibbs ripple.
- Amplitude law: harmonic n carries $1/n$ of the fundamental, so the edge's sharpness (which needs the cancellation of MANY harmonics) cannot survive a channel that stops at $n = 3$ – the output keeps ripple where the input has corners.