

2.5 Laplace transforms

Name: _____ Class: _____ Date: _____

Total: 23 marks

KEY WORDS Laplace transform

• partial fraction

• pole

Objective

Use the Laplace transform to turn ODEs into algebra, split rational functions by partial fractions, invert term by term, and read behaviour off the poles.

1 Key knowledge and worked examples

■ Three rules and one table

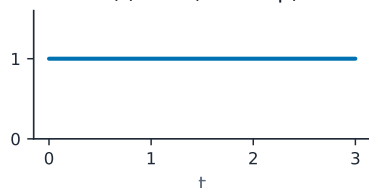
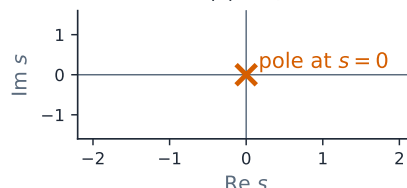
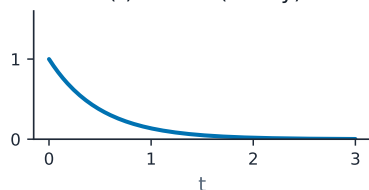
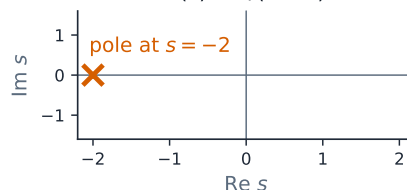
The **Laplace transform**

$$F(s) = \int_0^\infty f(t)e^{-st}dt \text{ converts differentiation}$$

into multiplication:

- $Ldy/dt = sY(s) - y(0)$ – the **initial condition** enters algebraically;
- $L\{1\} = 1/s$, $Le^{at} = 1/(s - a)$;
- **Partial fractions** split $F(s)$ into table-shaped pieces.

Each time curve maps to a pole; the pole position predicts the decay

 $f(t) = 1$ (the step) $F(s) = 1/s$  $f(t) = e^{-2t}$ (decay) $F(s) = 1/(s + 2)$ 

■ Worked example: the same ODE, the algebraic route

$$dy/dt + 2y = 10, y(0) = 1:$$

- Transform: $sY - 1 + 2Y = 10/s \rightarrow Y = (10/s + 1)/(s + 2)$.
- Split: $Y = 10/[s(s+2)] + 1/(s+2) = (5/s - 5/(s+2)) + 1/(s+2) = 5/s - 4/(s+2)$.
- Invert term by term: $y = 5 - 4e^{-2t}$ – identical to the classical route.

The identity $5/s - 4/(s+2) = [5(s+2) - 4s]/[s(s+2)] = (s + 10)/[s(s+2)]$ is the recombination check: split then recombine, and the fractions must return.

■ Poles predict behaviour

The **poles** of $Y(s)$ – the values of s that blow it up – are the exponents of the response: a pole at $s = 0$ is a constant; at $s = -a$, a decay e^{-at} ; a pair at $\pm j\omega$ is an oscillation. Read the poles and you have read the waveform before inverting anything.

2 Practice

2.1 Write $L\{3\}$, $L2e^{-5t}$, and $L\{d/dt f\}$ in terms of $F(s)$ and $f(0)$.

[3]

2.2 Split $6/[s(s+3)]$ into partial fractions and invert.

[4]

2.3 State the poles of $Y(s) = (2s + 6)/[(s+1)(s+3)]$ and name the waveform shape each implies.

[3]

3 Exam-style questions

3.1 Solve $dy/dt + 3y = 12$, $y(0) = 2$, entirely by Laplace: transform, solve for Y , split by partial fractions, invert. Verify against the classical route and by both endpoint checks. [7]

3.2 An RC network satisfies $dv/dt + 2v = 10u(t)$ with $v(0) = 0$, where $u(t)$ is the unit step.

Transform to $V(s)$, split by partial fractions, invert to $v(t)$, and name the pole locations and what each contributes to the response. Verify $v(0)$ directly from your time expression. [6]

Solutions

2.1

- $\mathcal{L}\{3\} = 3/s$.
- $\mathcal{L}2e^{-5t} = 2/(s+5)$.
- $\mathcal{L}df/dt = sF(s) - f(0)$.

2.2

- Split: $6/[s(s+3)] = A/s + B/(s+3)$.
- $A = 6/3 = 2$ (set $s = 0$); $B = -2$ (set $s = -3$).
- Invert: $2 - 2e^{-3t}$.

2.3

- Poles at $s = -1$ and $s = -3$.
- Each contributes a decaying exponential (e^{-t} , e^{-3t}) – a stable, non-oscillatory response.

3.1

- Transform: $sY - 2 + 3Y = 12/s \rightarrow Y = (12/s + 2)/(s+3)$.
- $Y = 12/[s(s+3)] + 2/(s+3)$.
- Split: $12/[s(s+3)] = 4/s - 4/(s+3)$, so $Y = 4/s - 2/(s+3)$.
- Invert: $y = 4 - 2e^{-3t}$.
- Classical: homogeneous Ce^{-3t} , particular $12/3 = 4$, $y(0) = 4 + C = 2 \rightarrow C = -2$: identical ✓.
- Checks: $y(0) = 2$ ✓; $y(\infty) = 4$, $0 + 3(4) = 12$ ✓.

3.2

- Transform: $sV + 2V = 10/s \rightarrow V = 10/[s(s+2)]$.
- Split: $V = 5/s - 5/(s+2)$.
- Invert: $v = 5(1 - e^{-2t})$ V.
- Poles at $s = 0$ (the constant 5 V final value) and $s = -2$ (the exponential settling with time constant 0.5 s).
- $v(0) = 5(1 - 1) = 0$ ✓.