

2.4 Ordinary differential equations

Name: _____ Class: _____ Date: _____

Total: 23 marks**KEY WORDS** ordinary differential equation

• homogeneous

• particular

• initial condition

Objective

Solve first-order linear ODEs with constant coefficients by the classical route (homogeneous + particular + initial condition) and verify each solution at both endpoints.

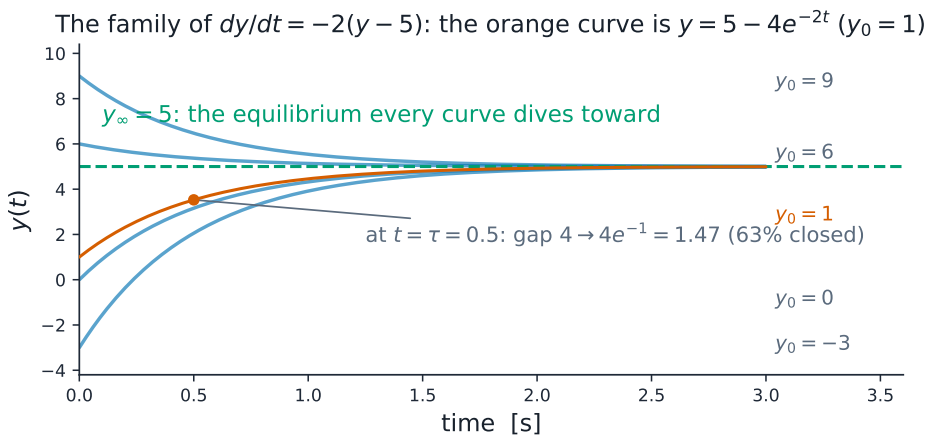
1 Key knowledge and worked examples

■ Homogeneous plus particular

A first-order linear **ODE** $dy/dt + ay = f(t)$ solves as the sum of:

- The **homogeneous** solution Ce^{-at} – the natural decay;
- A **particular** solution forced by $f(t)$ (a constant input forces a constant output: $y_p = f/a$).

The one **initial condition** $y(0)$ then fixes C .



■ Worked example, with both endpoint checks

$$dy/dt + 2y = 10, y(0) = 1.$$

- Homogeneous: Ce^{-2t} ; particular: $10/2 = 5$.
- General: $y = 5 + Ce^{-2t}$; $y(0) = 5 + C = 1$ gives $C = -4$.
- Solution: $y = 5 - 4e^{-2t}$.

Check 1 (start): $y(0) = 5 - 4 = 1 \checkmark$. Check 2 (steady state): as $t \rightarrow \infty$, $y \rightarrow 5$ and $dy/dt \rightarrow 0$; the equation reads $0 + 2(5) = 10 \checkmark$. Endpoint checks are free and catch most algebra slips.

■ It is the RC equation

$RCdv/dt + v = V_s$ rearranges exactly into this form – the wave-1 transient on a general footing. The method, not the circuit, is the content: any first-order system (thermal, fluid, mechanical) with a constant drive answers the same way, with time constant $\tau = 1/a$.

2 Practice

2.1 Solve $dy/dt + 3y = 6$, $y(0) = 2$. Check both endpoints. [3]

2.2 Solve $dy/dt + y = 0$, $y(0) = 5$, and state the time constant. [3]

2.3 For the ODE of the worked example ($y = 5 - 4e^{-2t}$), compute $y(0.5)$ and dy/dt at $t = 0.5$, and verify the ODE holds there. [4]

3 Exam-style questions

3.1 An RC circuit with $R = 2k\Omega$, $C = 500\mu F$ is driven by a 9 V step; the capacitor starts at 1 V.

Write the ODE for $v(t)$, solve it classically, and give the time constant, $v(0)$, $v(\infty)$, and v at one time constant. Verify the ODE at $t = 0$ numerically. [7]

3.2 A sensor cools according to $dT/dt = -k(T - 20)$ with $k = 0.1/min$, from $T(0) = 90^\circ C$.

Rearrange into standard form, solve, and find the time at which $T = 30^\circ C$. Check by substitution of your time back into the solution. [6]

Solutions

2.1

- $y = 2 + Ce^{-3t}$; $y(0) = 2 + C = 2$ gives $C = 0$.
- So $y = 2$ – the initial value already equals the steady value.
- Checks: $y(0) = 2 \checkmark$; $dy/dt = 0$, $0 + 3(2) = 6 \checkmark$.

2.2

- $y = Ce^{-t}$; $y(0) = 5 \rightarrow y = 5e^{-t}$.
- Time constant $\tau = 1$ (unit of time).

2.3

- $y(0.5) = 5 - 4e^{-1} = 5 - 1.472 = 3.528$.
- $dy/dt = 8e^{-2t}$; at 0.5: $8e^{-1} = 2.943$.
- ODE check: $2.943 + 2(3.528) = 2.943 + 7.056 = 9.999 \approx 10 \checkmark$.

3.1

- ODE: $RCdv/dt + v = 9 \rightarrow dv/dt + v/(RC) = 9/(RC)$ with $RC = 1s$.
- $dv/dt + v = 9$, so $v = 9 + Ce^{-t}$; $v(0) = 1$ gives $C = -8$: $v = 9 - 8e^{-t}$.
- $\tau = RC = 1s$; $v(0) = 1 \text{ V}$; $v(\infty) = 9 \text{ V}$.
- At $t = \tau$: $v = 9 - 8e^{-1} = 6.06V$.
- ODE check at $t = 0$: $dv/dt = 8e^{-t} = 8$, and $8 + 1 = 9 \checkmark$.

3.2

- Standard form: $dT/dt + 0.1T = 2$ (since $0.1 \times 20 = 2$).
- $T = 20 + Ce^{-0.1t}$; $T(0) = 90$ gives $C = 70$: $T = 20 + 70e^{-0.1t}$.
- Set $T = 30$: $70e^{-0.1t} = 10 \rightarrow e^{-0.1t} = 1/7$.
- $-0.1t = -\ln 7 = -1.946 \rightarrow t = 19.46min$.
- Check: $T(19.46) = 20 + 70e^{-1.946} = 20 + 70 \times 0.1429 = 30.0 \checkmark$.