2.3 Linear algebra essentials

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS linear system • matrix inverse • determinant • singular

Objective

Solve 2×2 and 3×3 linear systems by elimination, invert a 2×2 matrix, detect singularity from the determinant, and verify every solution by substitution into the unused equations.

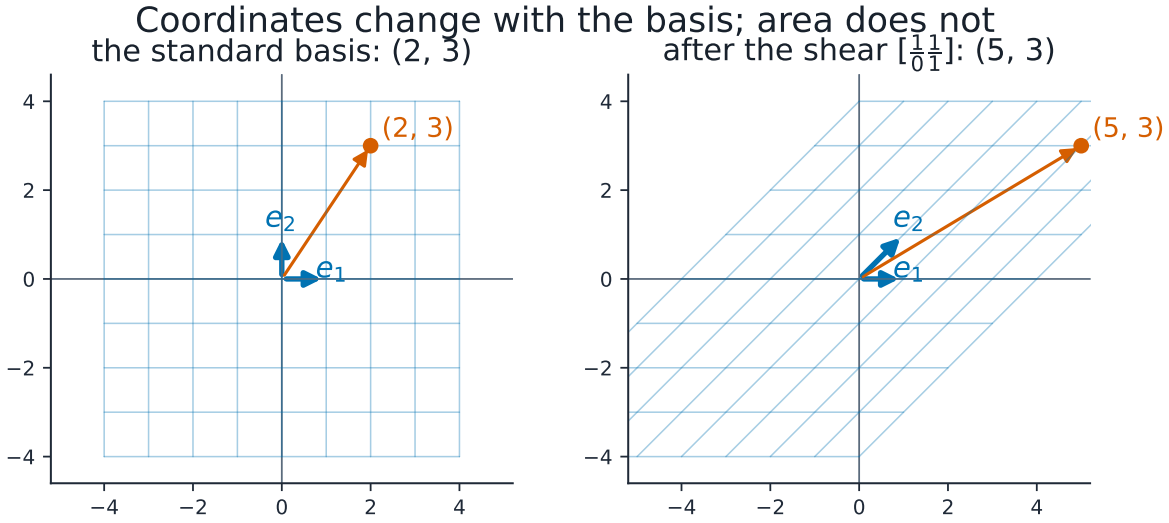
1 Key knowledge and worked examples

■ Elimination, and what the determinant says

A **linear system** `Ax = b` is solved by elimination: use one equation to remove a variable from another. The **determinant** of a 2×2 matrix,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$

is non-zero exactly when the rows are independent – a **singular** system (det = 0) has no unique solution: in circuits, a floating node or a loop of dependent sources.



■ Worked example: two routes, each the other's check

Solve $x + 2y = 8$, $3x + y = 9$.

- Elimination: row2 $-$ 3 $\cdot$ row1 $\rightarrow -5y = -15$, so $y = 3$, back-substitute $x = 2$.
- Inverse: $det = 1{\cdot}1 - 2{\cdot}3 = -5$; $A^{-1} = (1/-5)(1 - 2; -31)$, so $x = (8 - 2{\cdot}9)/(-5) = 2$, $y = (-24 + 9)/(-5) = 3$.

Substitution closes it: `2 + 2(3) = 8` ✓, `3(2) + 3 = 9` ✓. The **matrix inverse** route costs more arithmetic but verifies the elimination result with an independent method.

■ Scaling intuition for later courses

Nodal analysis IS `Ax = b`: `A` holds the conductances, `x` the node voltages, `b` the source currents. A well-posed circuit gives a non-singular `A`; a singular one means the modelling missed a ground or duplicated a source constraint – the determinant is a circuit-debugging tool, not just algebra.

2 Practice

2.1 Solve by elimination: $2x + y = 7$, $x - y = 2$. Verify by substitution. [3]

2.2 Compute the determinant of `(1 2; 3 1)` and use the inverse to solve $x + 2y = 8$, $3x + y = 9$. [4]

2.3 For which value of `k` is `(2 k; 4 6)` singular? Explain the geometric meaning in one sentence. [3]

3 Exam-style questions

3.1 A two-node circuit yields the system $3V_1 - V_2 = 12$, $-V_1 + 4V_2 = 0$ (conductances in siemens, currents in amps).

Solve for V_1 and V_2 by elimination, then verify by the determinant/inverse route. Finally compute the determinant of the coefficient matrix and state what it would mean for the circuit if it were zero. [7]

3.2 Mesh analysis of a two-loop network gives $5I_1 - 3I_2 = 12$ and $-3I_1 + 4I_2 = -6$.

Write the system in matrix form, compute the determinant, solve for both mesh currents, and check both original equations by substitution. State the physical meaning of $I_1 - I_2$. [6]

Solutions

2.1

- Adding the equations: $3x = 9$, $x = 3$.
- Then $y = 1$.
- Check: `2(3) + 1 = 7` ✓, `3 - 1 = 2` ✓.

2.2

- $det = 1 - 6 = -5$.
- $A^{-1} = (1/-5)(1 - 2; -31)$.
- $x = (8 - 18)/(-5) = 2$.
- $y = (-24 + 9)/(-5) = 3$.

2.3

- $det = 2 \cdot 6 - 4k = 12 - 4k = 0 \rightarrow k = 3$.
- With $k = 3$ the second row is twice the first: both equations describe the same line, so the system has no unique crossing point.

3.1

- From eq2: $V_2 = V_1/4$.
- Substitute: $3V_1 - V_1/4 = 12 \rightarrow (11/4)V_1 = 12 \rightarrow V_1 = 48/11 = 4.364V$.
- $V_2 = 1.091V$.
- Inverse route: $det = 3 \cdot 4 - (-1)(-1) = 11$; $V_1 = (12 \cdot 4 - (-1) \cdot 0)/11 = 48/11$ ✓, $V_2 = (3 \cdot 0 - (-1) \cdot 12)/11 = 12/11$ ✓.
- A zero determinant would mean the two node equations were dependent – the circuit model had no unique solution (e.g. a floating network with no ground).

3.2

- Matrix form: $\begin{pmatrix} 5 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \end{pmatrix}$.
- $det = 5 \cdot 4 - 9 = 11$.
- $I_1 = (12 \cdot 4 - (-3)(-6))/11 = (48 - 18)/11 = 30/11 = 2.727A$.
- $I_2 = (5 \cdot (-6) - (-3)(12))/11 = (-30 + 36)/11 = 6/11 = 0.545A$.
- Checks: `5(2.727) - 3(0.545) = 13.64 - 1.64 = 12.0` ✓; `-3(2.727) + 4(0.545) = -8.18 + 2.18 = -6.0` ✓.
- $I_1 - I_2$ is the current through the shared 3 Ω branch (downward through the shared element, by the loop orientation).