

KEY WORDS bilinear transform • prewarp

Prewarp cutoffs, transform analog prototypes, and verify the digital result at dc and at the cutoff.

- The bilinear transform

$s = \frac{2}{T} \cdot \frac{1-z^{-1}}{1+z^{-1}}$: left half-plane maps inside the unit circle (stability free), frequencies warp by $\omega_a = \frac{2}{T} \tan(\omega_d T/2)$ – prewarp every critical frequency.

- Worked example: the 100 Hz biquad

$f_s = 1$ kHz: prewarp $\omega_a = 2000 \tan(0.1\pi) = 649.8$ rad/s, $K = 1/\tan(0.1\pi) = 3.078$, and the 2nd-order Butterworth becomes $H(z) = \frac{1+z^{-2}}{14.82-16.94z^{-1}+6.12z^{-2}}$ ✓.

- Verify, then pay

DC gain $4/4 = 1.000 \checkmark$; $|H|$ at 100 Hz = -3.0 dB $\checkmark$ (cutoff placed exactly); poles at $|z| = 0.643 \checkmark$. Skip the prewarp ($K = 1$) and it cuts at 250 Hz – silently off-spec. The bill: 5 coefficients vs 67 taps, nonlinear phase, sensitive denominators.

2.1 Prewarp a 250 Hz cutoff at $f_s = 1$ kHz.

2.2 What does the bilinear transform do to the analog point $s = \sigma + j0$ with $\sigma < 0$? [3]

2.3 A digital filter has poles at $z = 0.9$ and $z = 0.5$. Stable? Why? [3]

3.1 Design a 1st-order Butterworth low-pass at 1 kHz cutoff, $f_s = 8$ kHz, by bilinear transform.

Prewarp the cutoff. Apply the transform with the prewarped constant to $H(s) = 1/(s+1)$ (s normalised by ω_a), giving $H(z)$ as $b_0(1+z^{-1})/(1+a_1z^{-1})$. Verify the dc gain and state the response at the cutoff frequency. [8]

3.2 The naive substitution ($K = 1$, no prewarp) is used for a 100 Hz, $f_s = 1$ kHz 2nd-order Butterworth.

State where the -3 dB point lands and why. Compute the response the naive filter shows at 100 Hz. Explain in one sentence why the error is small at low cutoffs and fatal near Nyquist. [7]

Solutions

2.1

- $\omega_a = 2000 \tan(\pi/4) = 2000 \text{ rad/s}$.

2.2

- Maps to a real z with $|z| < 1$: the whole left half-plane goes strictly inside the unit circle, so stable analog points stay stable digital points.

2.3

- Stable: both poles satisfy $|z| < 1$ – for rational systems BIBO stability is exactly all poles inside the unit circle.

3.1

- Prewarp: $\omega_a = 16\,000 \tan(\pi/8) = 6627 \text{ rad/s}$, $K = 1/\tan(\pi/8) = 2.414$.
- $s/\omega_a = K(1 - z^{-1})/(1 + z^{-1})$ in $H = 1/(s/\omega_a + 1)$: $H(z) = \frac{1+z^{-1}}{(K+1)+(1-K)z^{-1}}$.
- Numbers: $H(z) = \frac{1+z^{-1}}{3.414-1.414z^{-1}}$.
- DC gain $2/2 = 1.000 \checkmark$; at 1 kHz the prewarp places -3.01 dB exactly $\checkmark$.

3.2

- Lands at 250 Hz: the -3 dB point sits where $\tan(\omega T/2) = 1$, i.e. $f = f_s/4$.
- At 100 Hz the naive filter reads -0.05 dB – a complacent passband number that hides the miss.
- Near dc $\tan \theta \approx \theta$ so warp vanishes; near Nyquist the tangent compresses violently, so unwarped designs land far off.