12.3 The FFT

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS FFT • butterfly

Objective

Count FFT operations, map bins to hertz, and check real-time feasibility with operation counts.

1 Key knowledge and worked examples

The butterfly

Even/odd split + twiddle factors: $\log_2 N$ stages of $N/2$ butterflies. Counting complex operations: direct N^2, FFT $N \log_2 N$ – the gap widens with N.

Worked example: the count and the clock

$N = 1024$: $N^2 = 1\,048\,576$ against $N \log_2 N = 10\,240$ – 102× fewer ✓. At 10 ns per operation the transform takes about 100 μs – real time against a 1.024 s record.

Bin to hertz

Bin k sits at $f = k \cdot f_s/N$. Bin 100 of the $N = 1024$, $f_s = 1$ kHz transform is 97.66 Hz ✓ – reading bin numbers as hertz is the classic lab error.

2 Practice

2.1 How many stages and butterflies for an $N = 64$ radix-2 FFT? [3]

2.2 At $f_s = 8$ kHz with $N = 512$: bin width, and the frequency of bin 90. [3]

2.3 Direct DFT of $N = 256$: operation count, and the FFT count. [3]

3 Exam-style questions

3.1 A real-time analyser samples at $f_s = 48$ kHz and must update every 21.3 ms (1024-sample hop).

Compute the bin width. Compute the FFT operation count and, at 5 ns per operation, the compute time per transform; state the margin against the hop. Compute which bin a 1 kHz tone lands in (and its exact frequency). [8]

3.2 A colleague proposes replacing an $N = 1024$ FFT by two $N = 512$ FFTs plus 512 combining butterflies to halve the latency.

Check the operation count of the proposal against one $N = 1024$ FFT. Explain where the saving (if any) comes from, and what Danielson–Lanczos says the structure actually is. State one reason the split helps pipeline implementations. [7]

Solutions

2.1

- $\log_2 64 = 6$ stages, each $N/2 = 32$ butterflies.

2.2

- $\Delta f = 8000/512 = 15.625$ Hz.
- Bin 90: $90 \times 15.625 = 1406.25$ Hz.

2.3

- Direct $256^2 = 65\,536$; FFT $256 \times 8 = 2048$ – 32× fewer.

3.1

- $\Delta f = 48\,000/1024 = 46.875$ Hz.
- FFT count $1024 \times 10 = 10\,240$ operations.
- At 5 ns: 51.2 μs per transform.
- Margin: 51.2 μs against 21.3 ms – about 400× headroom ✓.
- 1 kHz: $1000/46.875 = 21.33 \rightarrow$ bin 21, exact frequency $21 \times 46.875 = 984.375$ Hz (0.984 kHz).

3.2

- Two 512-FFTs: $2 \times 512 \times 9 = 9216$, plus 512 butterflies = 9728 operations – versus 10 240: a 5% saving only.
- Danielson–Lanczos: one 1024 DFT IS two 512 DFTs plus N twiddle combines – the proposal reinvents the first FFT stage.
- Splitting helps pipelining: shorter transforms mean smaller, shallower butterfly arrays with cleaner memory locality.