

12.2 DTFT and DFT

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS DFT • leakage

Objective

Compute small DFTs by hand, predict leakage from window choice, and turn resolution requirements into record lengths.

1 Key knowledge and worked examples

■ From DTFT to DFT

$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$ – the DTFT sampled at N points; sampling in frequency makes the time signal periodic, which is why off-bin tones leak.

■ Worked example: on-bin versus off-bin

8-point DFT of a cosine on bin 1: $|X| = 4$ at bins 1 and 7, zero elsewhere ✓. Half a bin off ($1.5 \cdot f_s/8$): magnitudes 2.40, 2.80, 1.19, 1.0 – energy in every bin. The rectangular window’s -13 dB sidelobes are speaking; Hamming drops them to -43 dB.

■ The resolution budget

$\Delta f = f_s/N$: at $f_s = 1$ kHz, $N = 100$ gives 10 Hz bins, $N = 1000$ gives 1 Hz. Zero-padding interpolates the same spectrum – it resolves nothing new.

2 Practice

2.1 At $f_s = 2$ kHz, what N gives 4 Hz bins? [3]

2.2 A tone sits at bin 3 of an on-bin 8-point DFT with amplitude 2. Give $|X[k]|$ for all bins. [3]

2.3 Two tones 3 Hz apart must be resolved at $f_s = 1$ kHz. Minimum N , and why padding to 8192 does not help. [3]

3 Exam-style questions

3.1 A spectrum analyser at $f_s = 1$ kHz must resolve the 50 Hz mains tone from a 55 Hz interferer.

State the minimum N and the record length in seconds. Compute the bin width for $N = 256$ and state whether the two tones are resolvable. Explain what zero-padding to 2048 changes and what it cannot change. [7]

3.2 An 8-point DFT of $x[n] = \cos(2\pi \times 1.5 \times n/8)$ is computed with a rectangular window.

Explain why every bin is non-zero. Give the two largest bins and their magnitudes. The same tone is now windowed with Hamming: state what happens to the sidelobe level and to the main-lobe width, and the consequence for resolving two nearby tones. [8]

Solutions

2.1

- $N = f_s/\Delta f = 2000/4 = 500$.

2.2

- Amplitude 2 $\rightarrow$ each exponential carries amplitude 1, summing over N samples:
 $|X[3]| = |X[5]| = A N/2 = 2 \times 4 = 8$, all other bins 0.

2.3

- $\Delta f \leq 3$ Hz needs $N \geq 1000/3 = 334 \rightarrow N = 334$ or more (0.334 s).
- Padding only interpolates the same spectrum – no new information to separate the tones.

3.1

- $\Delta f < 5$ Hz separation needs $N > 1000/5 = 200$: minimum $N = 200$ (0.2 s).
- $N = 256$: $\Delta f = 3.906$ Hz.
- $3.906 < 5 \rightarrow$ resolvable $\checkmark$ (bins $50/3.906 = 12.8$ and $55/3.906 = 14.08$ – different bins).
- Padding to 2048 draws the same underlying spectrum with $8\times$ more points (prettier plot); it cannot create the frequency separation that the 0.2 s record already fixed.

3.2

- Sampling in frequency $\Rightarrow$ periodic time: a 1.5-bin tone's periodic copies do not align, so the mismatch smears into every bin – leakage.
- Largest: $|X[2]| = |X[6]| = 2.80$, then $|X[1]| = |X[7]| = 2.40$.
- Hamming: sidelobes drop from -13 dB to -43 dB (small tones beside large ones reappear).
- Main lobe widens (about $2\times$ the rectangle): two nearby tones now blur together – the trade is dynamic range against resolution.