

12.1 Discrete-time signals and systems review

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS impulse response • convolution

Objective

Compute discrete convolutions by hand, read stability and causality off an impulse response, and connect poles to settling behaviour.

1 Key knowledge and worked examples

■ **The sliding table**

$y[n] = \sum_k h[k] \cdot x[n - k]$: slide the taps, add one diagonal at a time. A step through any FIR settles at $\sum h$ – the DC gain $H(1)$.

■ **Worked example: two smoothers**

$h = [1, 2, 1]$ on a unit step: $y = 1, 3, 4, 4, \dots$ ✓ settles at 4. The recursive averager $y[n] = (y[n - 1] + x[n])/2$ steps up 0.5, 0.75, 0.875, 0.9375 = $1 - 2^{-n}$ – its pole sits at $z = 0.5$, so it approaches but never arrives.

■ **Stability and causality**

Causality: $h[n] = 0$ for $n < 0$. BIBO stability: $\sum |h[n]|$ converges – for rational systems, all poles inside the unit circle.

2 Practice

2.1 Compute $y[n]$ for $h = [1, 3, 2]$ and $x = [1, 1, 1]$. [3]

2.2 A filter has $h[n] = 0.4^n$ for $n \geq 0$. Is it BIBO stable? Justify with the sum. [3]

2.3 Give the DC gain of $h = [2, -1, 2]$ and check it against the steady step output. [3]

3 Exam-style questions

3.1 A 3-tap FIR has $h = [1, a, 1]$.

For $a = 2$ compute the step response and its settling value. For which a does the step response stay non-negative at every sample? State the causality and BIBO status of this filter, with reasons. [7]

3.2 The leaky integrator $y[n] = \alpha y[n-1] + x[n]$ with $y[-1] = 0$ and $0 < \alpha < 1$.

Compute the impulse response, the step response at $n = 0, 1, 2, 3$ for $\alpha = 0.5$, and its closed form. State the pole, the stability condition on α , and how many samples until the step response is within 1% of its limit when $\alpha = 0.5$. [8]

Solutions

2.1

- Diagonals: $y = 1, 4, 6, 5, 2$.

2.2

- $\sum 0.4^n = 1/(1 - 0.4) = 1.667$ converges ✓ stable.

2.3

- $H(1) = 2 - 1 + 2 = 3$; steady step output 3 ✓.

3.1

- $a = 2$: step response $1, 3, 4, 4, \dots$, settling at $\sum h = 4$.
- Non-negative everywhere needs $y[1] = 1 + a \geq 0$ and $y[3] = 2 + a \geq 0$: the binding sample is $n = 1$, so $a \geq -1$.
- Causal: $h[n] = 0$ for $n < 0$ ✓ (finite taps).
- BIBO: finite $\sum |h| = |1| + |a| + |1|$ always converges ✓ every FIR is unconditionally stable.

3.2

- Impulse response $h[n] = \alpha^n, n \geq 0$.
- Step: $1, 1.5, 1.75, 1.875$ ✓.
- Closed form $y[n] = (1 - \alpha^{n+1})/(1 - \alpha)$; at $\alpha = 0.5$: $2 - 2^{-n}$ ✓ (limit 2).
- Pole at $z = \alpha$; stable for $0 < \alpha < 1$ ($\sum \alpha^n$ converges).
- Within 1% of limit 2: remaining gap $2^{-n} \leq 0.02 \Rightarrow n \geq 6$ samples.