

10.8 State-space introduction

Name: _____ Class: _____ Date: _____ **Total: 24 marks****KEY WORDS** state space • transition matrix • pole placement

Objective

Convert between transfer-function and state-space forms, propagate the state with the transition matrix, and design pole placement by full-state feedback.

1 Key knowledge and worked examples

■ The two forms

$G(s) = C(sI - A)^{-1}B + D \leftrightarrow x' = Ax + Bu, y = Cx + Du$. Companion form for $b/(s^2 + a_1s + a_0)$: $A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ – eigenvalues of A are the poles, always.

■ Worked example: the transition matrix

For $A = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}$ (poles $-2, -5$): the series $\Phi(t) = I + At + (At)^2/2! + \dots$ evaluated at $t = 0.3$ gives $\Phi = \begin{bmatrix} 0.766 & 0.109 \\ -1.086 & 0.006 \end{bmatrix}$, whose eigenvalues are $e^{-0.6} = 0.549$ and $e^{-1.5} = 0.223$ – the poles made real ✓.

■ Pole placement

Full-state feedback $u = -Kx$ moves the eigenvalues of $A - BK$ wherever the (controlable) system allows: for $s^2 + 20s + 96$ from $A = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: matching coefficients gives $K = [86, 13]$ – eigenvalues compute to $-8, -12$ ✓.

2 Practice

2.1 Write state-space (scalar) for $G = 2/(s + 3)$. [3]

2.2 Give the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$. [3]

2.3 State $\Phi(0)$ and give $x(t)$ for $x(0)$ when the input is zero. [3]

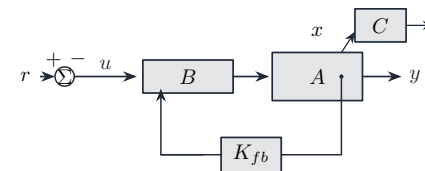
3 Exam-style questions

3.1 Realise $G(s) = 10/(s^2 + 7s + 10)$ in companion form.

Write A, B and C, and verify the eigenvalues against the poles. Verify the transfer function is recovered from $C(sI - A)^{-1}B$. Compute $\Phi(0.3)$ by the series and give $x(0.3)$ for $x(0) = [1, 0]$ with $u = 0$. Check $x_1(0.3)$ against the closed form $(5/3)e^{-0.6} - (2/3)e^{-1.5}$. [8]

3.2 For the A, B above, design full-state feedback placing both poles at -8 and -12 .

Write the closed-loop characteristic polynomial of $A - BK$ in terms of k_1, k_2 . Match $s^2 + 20s + 96$ and give K. Verify the eigenvalues. State the two practical conditions (one on the plant, one on the instrumentation) that must hold for this design to run on real hardware. [7]



Solutions

2.1

- $x' = -3x + 2u$, $y = x$.

2.2

- $\lambda^2 + 4\lambda + 4 = 0 \rightarrow$ both -2 (repeated).

2.3

- $\Phi(0) = I$ (zeroth series term).
- $x(t) = \Phi(t)x(0)$.

3.1

- $A = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 10 & 0 \end{bmatrix}$.
- $\det(\lambda I - A) = \lambda^2 + 7\lambda + 10 \rightarrow$ eigenvalues $-2, -5 =$ the poles $\checkmark$.
- $C(sI - A)^{-1}B$: adjugate route $-(sI - A)^{-1}B = \begin{bmatrix} 1 \\ s \end{bmatrix} / (s^2 + 7s + 10)$; $C \cdot that = 10 / (s^2 + 7s + 10)$ $\checkmark$ recovered.
- Series at $t = 0.3$: $\Phi = \begin{bmatrix} 0.766 & 0.109 \\ -1.086 & 0.006 \end{bmatrix}$.
- $x(0.3) = \Phi(0.3)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.766 \\ -1.086 \end{bmatrix}$.
- Closed form: $(5/3)(0.549) - (2/3)(0.223) = 0.915 - 0.149 = 0.766$ $\checkmark$ both routes agree.

3.2

- $A - BK = \begin{bmatrix} 0 & 1 \\ -10 - k_1 & -7 - k_2 \end{bmatrix} \rightarrow$ char $s^2 + (7 + k_2)s + (10 + k_1)$.
- Match $s^2 + 20s + 96$: $k_1 = 86$, $k_2 = 13$.
- Eigenvalues of $A - BK$ compute to $-8, -12$ $\checkmark$.
- Plant condition: controllability (here $[BAB]$ has rank 2 – the coefficients k_1, k_2 were free to match, which is what controllability buys).
- Instrumentation condition: the FULL state must be available – every x measured, or an observer/estimator built (observable) and its estimate fed back.