10.8 State-space introduction

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS state space • transition matrix • pole placement

Objective

Convert between transfer-function and state-space forms, propagate the state with the transition matrix, and design pole placement by full-state feedback.

1 Key knowledge and worked examples

■ The two forms

$G(s) = C(sI - A)^{-1}B + D \leftrightarrow x' = Ax + Bu, y = Cx + Du$. Companion form for $b/(s^2 + a_1 s + a_0)$: A = [[0,1],[−a$_0$,−a$_1$]], B = [0;1] – eigenvalues of A are the poles, always.

■ Worked example: the transition matrix

For A = [[0,1],[−10,−7]] (poles −2, −5): the series $\Phi(t) = I + At + (At)^2/2! + \cdots$ evaluated at t = 0.3 gives $\Phi = [[0.766, 0.109], [-1.086, 0.006]]$, whose eigenvalues are $e^{-0.6} = 0.549$ and $e^{-1.5} = 0.223$ – the poles made real ✓.

■ Pole placement

Full-state feedback $u = -Kx$ moves the eigenvalues of $A - BK$ wherever the (controllable) system allows: for $s^2 + 20s + 96$ from A = [[0,1],[−10,−7]], B = [0;1]: matching coefficients gives $K = [86, 13]$ – eigenvalues compute to −8, −12 ✓.

2 Practice

2.1 Write state-space (scalar) for $G = 2/(s + 3)$. [3]

2.2 Give the eigenvalues of A = [[0,1],[−4,−4]]. [3]

2.3 State $\Phi(0)$ and give x(t) for x(0) when the input is zero. [3]

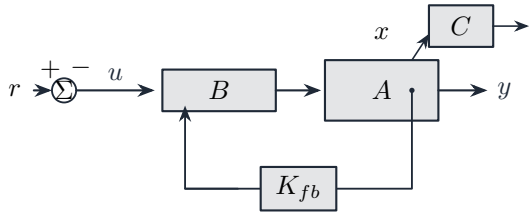
3 Exam-style questions

3.1 Realise $G(s) = 10/(s^2 + 7s + 10)$ in companion form.

Write A, B and C, and verify the eigenvalues against the poles. Verify the transfer function is recovered from $C(sI - A)^{-1}B$. Compute $\Phi(0.3)$ by the series and give x(0.3) for x(0) = [1, 0] with u = 0. Check $x_1(0.3)$ against the closed form $(5/3)e^{-0.6} - (2/3)e^{-1.5}$. [8]

3.2 For the A, B above, design full-state feedback placing both poles at -8 and -12.

Write the closed-loop characteristic polynomial of $A - BK$ in terms of k_1, k_2. Match $s^2 + 20s + 96$ and give K. Verify the eigenvalues. State the two practical conditions (one on the plant, one on the instrumentation) that must hold for this design to run on real hardware. [7]



Solutions

2.1

- $x' = -3x + 2u$, $y = x$.

2.2

- $\lambda^2 + 4\lambda + 4 = 0 \rightarrow$ both -2 (repeated).

2.3

- $\Phi(0) = I$ (zeroth series term).
- $x(t) = \Phi(t)x(0)$.

3.1

- A = [[0,1],[−10,−7]], B = [0; 1], C = [10, 0].
- $det(\lambda I - A) = \lambda^2 + 7\lambda + 10 \rightarrow$ eigenvalues -2, -5 = the poles ✓.
- $C(sI - A)^{-1}B$: adjugate route – $(sI - A)^{-1}B = [1; s]/(s^2 + 7s + 10)$; $C \cdot that = 10/(s^2 + 7s + 10)$ ✓ recovered.
- Series at t = 0.3: $\Phi = [[0.766, 0.109], [-1.086, 0.006]]$.
- $x(0.3) = \Phi(0.3)[1, 0] = [0.766, -1.086]$.
- Closed form: $(5/3)(0.549) - (2/3)(0.223) = 0.915 - 0.149 = 0.766$ ✓ both routes agree.

3.2

- $A - BK = [[0, 1], [-10 - k_1, -7 - k_2]] \rightarrow$ char $s^2 + (7 + k_2)s + (10 + k_1)$.
- Match $s^2 + 20s + 96$: $k_1 = 86$, $k_2 = 13$.
- Eigenvalues of $A - BK$ compute to -8, -12 ✓.
- Plant condition: controllability (here $[BAB]$ has rank 2 – the coefficients k_1, k_2 were free to match, which is what controllability buys).
- Instrumentation condition: the FULL state must be available – every x measured, or an observer/estimator built (observable) and its estimate fed back.