

# 10.5 Root locus

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 24 marks

KEY WORDS root locus • breakaway point

## Objective

Sketch root loci with the standard rules, design gain against a damping specification, and predict where closed-loop poles migrate.

## 1 Key knowledge and worked examples

### ■ The rules that build the sketch

Locus exists left of an odd count of real poles+zeros;  $n-m$  branches go to asymptotes  $180^\circ(2k+1)/(n-m)$  centred on the centroid  $\sigma = (\sum p - \sum z)/(n-m)$ ; branches meet at the breakaway points where  $\frac{d}{ds}[\text{denominator/numerator}] = 0$ .

### ■ Worked example: the three-pole plant

$K/[s(s+4)(s+6)]$ : centroid  $-3.33$ , asymptotes  $\pm 60^\circ/180^\circ$ ; breakaways at  $-1.57$  ( $K = 16.9$ ) and  $-5.10$ ; Routh says stable to  $K = 240$  – the locus crosses into the RHP at  $\omega = \sqrt{24} = 4.90 \text{ rad/s}$  at exactly that gain.

### ■ Design on a damping ray

For  $\zeta = 0.5$  the pole must lie on  $s = -0.5\omega + j0.866\omega$  where the angle condition holds; on  $K/[s(s+4)(s+6)]$  the point is  $-1.20 + j2.08$  with  $K = 43.8$  – read the transient specs straight off the point.

## 2 Practice

2.1 Sketch facts for  $K/[(s+1)(s+3)]$ : give the asymptote angles, the centroid, and the breakaway point with its  $K$ . [3]

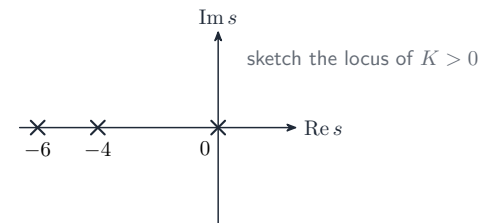
2.2 For  $K(s+3)/(s+1)$ : where does the second branch go (there is one zero at  $-3$ ), and does a breakaway point exist on the real axis? [3]

2.3 For  $K/[s(s+3)]$ , verify that  $s = -1.50 + j1.53$  satisfies the angle condition for  $\zeta = 0.7$ , and compute the  $K$  that places a closed-loop pole there. [3]

## 3 Exam-style questions

3.1 A unity-feedback system has open loop  $K/[s(s+4)(s+6)]$ .

Draw the real-axis segments that carry locus and compute the centroid and asymptote angles. Compute both breakaway points and the gain at the one between 0 and  $-4$ . Use Routh to find the gain and frequency at which the locus crosses the imaginary axis. Assemble the sketch order: which specification (damping or speed) does moving along the locus from the breakaway towards the asymptotes trade against which? [8]



**3.2** The plant  $K/[s(s+6)]$  is stable for all  $K$  but poorly damped at high gain. Consider instead adding a zero – control  $C(s)G(s) = K(s+2)/[s(s+6)]$  under unity feedback.

Write the closed-loop characteristic polynomial with unity feedback and show it is stable for ALL  $K > 0$  (Routh first column). Compute the closed-loop poles at  $K = 6$ . Explain, in one sentence each, what the added zero did to the far branch and to the near branch. [7]

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## Solutions

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### 2.1

- $n-m = 2 \rightarrow$  asymptotes  $\pm 90^\circ$ .
- Centroid  $\sigma = (-1 - 3)/2 = -2$ .
- Breakaway midway where  $d/ds[(s+1)(s+3)] = 2s+4 = 0 \rightarrow s = -2$ ,  $K = |(-1)(1)| = 1$ .

### 2.2

- One branch ends at the zero  $-3$ ; the other runs to  $-\infty$  along the real axis ( $n-m = 1$ , asymptote  $180^\circ$ ).
- $d/ds[(s+1)/(s+3)] = 2/(s+3)^2 \neq 0$  – no breakaway/break-in point.

### 2.3

- Angles from 0 and  $-3$ :  $\angle s = 180^\circ - 45.5^\circ = 134.5^\circ$ ,  $\angle(s+3) = 45.5^\circ$ ; sum  $180^\circ$  ✓ on the ray ( $\zeta = 0.7$  has imaginary/real ratio  $\sqrt{0.51}/0.7 = 1.02$  ✓ matching  $1.53/1.50$ ).
- $K = |s||s+3| = 2.14 \times 2.14 = 4.59$ .

### 3.1

- Real-axis locus on  $(-\infty, -6]$ ,  $[-4, 0]$ .
- Centroid  $\sigma = (0 - 4 - 6)/3 = -3.33$ ; asymptotes  $60^\circ, 180^\circ, -60^\circ$ .
- Breakaways:  $3s^2 + 20s + 24 = 0 \rightarrow s = -1.57$  and  $-5.10$ ; at  $-1.57$ ,  $K = |-1.57 \times 2.43 \times 4.43| = 16.9$ .
- Routh on  $s^3 + 10s^2 + 24s + K$ : stable for  $K < 240$ ; at  $K = 240$  the auxiliary  $10s^2 + 240 = 0$  gives  $s = \pm j\sqrt{24} = \pm j4.90$ .
- Trade: moving up the locus from the breakaway, damping falls as the poles leave the  $\zeta$  rays (overshoot grows) while  $\omega_n$  and speed rise – the classic speed-versus-damping exchange, ending in instability at  $K = 240$ .

### 3.2

- Characteristic:  $s^2 + (6 + K)s + 2K$ .
- Routh first column: 1,  $6+K$ ,  $2K$  – positive for every  $K > 0$ .
- At  $K = 6$ :  $s^2 + 12s + 12 \rightarrow s = -1.10, -10.90$ .
- The zero captured one branch: instead of heading to  $-\infty$  along an asymptote it terminates at  $-2$ , bending the locus left.
- The near branch now pairs with a FAR pole ( $-10.9$ ): the pair behaves like a single dominant pole at  $-1.10$  – the zero pulled the far branch away, cleaning the dominant dynamics.