

10.5 Root locus

Name: _____ Class: _____ Date: _____

Total: 24 marks

KEY WORDS root locus • breakaway point

Objective

Sketch root loci with the standard rules, design gain against a damping specification, and predict where closed-loop poles migrate.

1 Key knowledge and worked examples

■ The rules that build the sketch

Locus exists left of an odd count of real poles+zeros; $n-m$ branches go to asymptotes $180^\circ(2k+1)/(n-m)$ centred on the centroid $\sigma = (\sum p - \sum z)/(n-m)$; branches meet at the breakaway points where $\frac{d}{ds}[\text{denominator/numerator}] = 0$.

■ Worked example: the three-pole plant

$K/[s(s+4)(s+6)]$: centroid -3.33 , asymptotes $\pm 60^\circ/180^\circ$; breakaways at -1.57 ($K = 16.9$) and -5.10 ; Routh says stable to $K = 240$ – the locus crosses into the RHP at $\omega = \sqrt{24} = 4.90 \text{ rad/s}$ at exactly that gain.

■ Design on a damping ray

For $\zeta = 0.5$ the pole must lie on $s = -0.5\omega + j0.866\omega$ where the angle condition holds; on $K/[s(s+4)(s+6)]$ the point is $-1.20 + j2.08$ with $K = 43.8$ – read the transient specs straight off the point.

2 Practice

2.1 Sketch facts for $K/[(s+1)(s+3)]$: give the asymptote angles, the centroid, and the breakaway point with its K . [3]

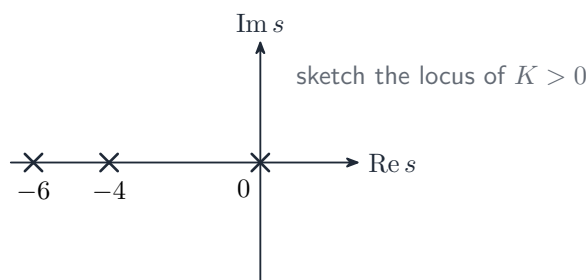
2.2 For $K(s+3)/(s+1)$: where does the second branch go (there is one zero at -3), and does a breakaway point exist on the real axis? [3]

2.3 For $K/[s(s+3)]$, verify that $s = -1.50 + j1.53$ satisfies the angle condition for $\zeta = 0.7$, and compute the K that places a closed-loop pole there. [3]

3 Exam-style questions

3.1 A unity-feedback system has open loop $K/[s(s+4)(s+6)]$.

Draw the real-axis segments that carry locus and compute the centroid and asymptote angles. Compute both breakaway points and the gain at the one between 0 and -4 . Use Routh to find the gain and frequency at which the locus crosses the imaginary axis. Assemble the sketch order: which specification (damping or speed) does moving along the locus from the breakaway towards the asymptotes trade against which? [8]



3.2 The plant $K/[s(s+6)]$ is stable for all K but poorly damped at high gain. Consider instead adding a zero – control $C(s)G(s) = K(s+2)/[s(s+6)]$ under unity feedback.

Write the closed-loop characteristic polynomial with unity feedback and show it is stable for ALL $K > 0$ (Routh first column). Compute the closed-loop poles at $K = 6$. Explain, in one sentence each, what the added zero did to the far branch and to the near branch. [7]

Solutions

2.1

- $n-m = 2 \rightarrow$ asymptotes $\pm 90^\circ$.
- Centroid $\sigma = (-1 - 3)/2 = -2$.
- Breakaway midway where $d/ds[(s+1)(s+3)] = 2s+4 = 0 \rightarrow s = -2$, $K = |(-1)(1)| = 1$.

2.2

- One branch ends at the zero -3 ; the other runs to $-\infty$ along the real axis ($n-m = 1$, asymptote 180°).
- $d/ds[(s+1)/(s+3)] = 2/(s+3)^2 \neq 0$ – no breakaway/break-in point.

2.3

- Angles from 0 and -3 : $\angle s = 180^\circ - 45.5^\circ = 134.5^\circ$, $\angle(s+3) = 45.5^\circ$; sum 180° ✓ on the ray ($\zeta = 0.7$ has imaginary/real ratio $\sqrt{0.51}/0.7 = 1.02$ ✓ matching $1.53/1.50$).
- $K = |s||s+3| = 2.14 \times 2.14 = 4.59$.

3.1

- Real-axis locus on $(-\infty, -6]$, $[-4, 0]$.
- Centroid $\sigma = (0 - 4 - 6)/3 = -3.33$; asymptotes 60° , 180° , -60° .
- Breakaways: $3s^2 + 20s + 24 = 0 \rightarrow s = -1.57$ and -5.10 ; at -1.57 , $K = |-1.57 \times 2.43 \times 4.43| = 16.9$.
- Routh on $s^3 + 10s^2 + 24s + K$: stable for $K < 240$; at $K = 240$ the auxiliary $10s^2 + 240 = 0$ gives $s = \pm j\sqrt{24} = \pm j4.90$.
- Trade: moving up the locus from the breakaway, damping falls as the poles leave the ζ rays (overshoot grows) while ω_n and speed rise – the classic speed-versus-damping exchange, ending in instability at $K = 240$.

3.2

- Characteristic: $s^2 + (6+K)s + 2K$.
- Routh first column: 1, $6+K$, $2K$ – positive for every $K > 0$.
- At $K = 6$: $s^2 + 12s + 12 \rightarrow s = -1.10, -10.90$.
- The zero captured one branch: instead of heading to $-\infty$ along an asymptote it terminates at -2 , bending the locus left.
- The near branch now pairs with a FAR pole (-10.9): the pair behaves like a single dominant pole at -1.10 – the zero pulled the far branch away, cleaning the dominant dynamics.