

10.3 Time-domain response

Name: _____ Class: _____ Date: _____ **Total: 24 marks****KEY WORDS** overshoot • settling time • steady-state error

Objective

Compute step-response specifications from the second-order prototype, map pole locations to transient behaviour, and read steady-state error constants off the loop.

1 Key knowledge and worked examples

■ The prototype formulas

$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$, $t_p = \pi/(\omega_n\sqrt{1-\zeta^2})$, $t_s \approx 4/(\zeta\omega_n)$ (2% band). Poles at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$: real part decays, imaginary part rings, distance from origin is ω_n .

■ Worked example: one number, three specs

$\omega_n = 6$, $\zeta = 0.4$: $M_p = 25.4\%$, $t_p = 0.571s$, $t_s = 4/2.4 = 1.67s$ – and poles at $-2.4 \pm j5.51$ (check: $|pole| = \sqrt{2.4^2 + 5.51^2} = 6 \checkmark$).

■ Error constants

Type 0: $K_p = G(0)$, step error $1/(1 + K_p)$. Type 1: $K_v = \lim_{s \rightarrow 0} sG$, ramp error $1/K_v$; step error zero. Each integrator buys one order of tracking and costs phase.

2 Practice

2.1 A system has $\omega_n = 6 \text{ rad/s}$, $\zeta = 0.4$. Compute M_p , t_p and t_s (2%). [3]

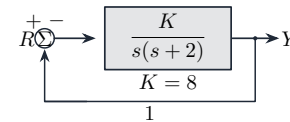
2.2 Poles at $-2 \pm j4$: compute ω_n , ζ and state whether the step response overshoots. [3]

2.3 Compute the dc gain and the unit-step steady-state error for $G = 5/(s+2)$ in unity feedback. [3]

3 Exam-style questions

3.1 A unity-feedback system has loop $G = K/[s(s+2)]$ with $K = 8$.

Write the closed-loop transfer function and its ω_n and ζ . Compute M_p , t_p and t_s . Compute the unit-ramp steady-state error via K_v . Determine the K that makes $\zeta = 0.5$ exactly, and verify. [8]



3.2 A servo must track a ramp with error at most 0.1 and settle (2%) within 2 s. The plant is $G = K/[s(s + 2)]$.

Derive the ramp-error requirement on K_v and hence the minimum K . Derive the settling-time requirement on $\zeta\omega_n$ and check it at that K . If the two requirements conflict, state which gain each wants and propose the standard resolution (a compensator, not a compromise gain). [7]

Solutions

2.1

- $M_p = e^{-\pi \cdot 0.4 / 0.9165} = 25.4\%$.
- $t_p = \pi / (6 \times 0.9165) = 0.571s$.
- $t_s = 4 / (0.4 \times 6) = 1.67s$.

2.2

- $\omega_n = |pole| = \sqrt{4 + 16} = 4.47rad/s$.
- $\zeta = 2/4.47 = 0.447$ – underdamped, so yes it overshoots ($M_p \approx 20.8\%$).

2.3

- $K_p = G(0) = 5/2 = 2.5$.
- $e_{ss} = 1/(1 + K_p) = 1/3.5 = 0.286$.

3.1

- $T = 8/(s^2 + 2s + 8)$: $\omega_n = 2.83rad/s$, $\zeta = 2/(2 \times 2.83) = 0.354$.
- $M_p = e^{-\pi \cdot 0.354 / 0.935} = 30.5\%$; $t_p = \pi / (2.83 \times 0.935) = 1.19s$; $t_s = 4 / (0.354 \times 2.83) = 4.0s$.
- $K_v = \lim_{s \rightarrow 0} sG(s) = K/2 = 4 \rightarrow$ ramp error $1/4 = 0.25$.
- $\zeta = 0.5$ with the s coefficient fixed at 2: $\omega_n = \sqrt{K}$ and $\zeta = 2/(2\sqrt{K})$, so $\zeta = 0.5$ needs $\sqrt{K} = 2$, giving $K = 4$. Check: $s^2 + 2s + 4 \rightarrow \zeta = 2/(2 \times 2) = 0.5$ exactly.

3.2

- Ramp error $= 1/K_v = 2/K \leq 0.1 \rightarrow K \geq 20$.
- $\zeta\omega_n = 2/2 = 1$ always (the s coefficient is 2) $\rightarrow t_s = 4/1 = 4s$ at ANY gain of this plant.
- At $K = 20$ the ramp spec holds but settling is $4s > 2s$ – and raising K only worsens damping ratio ($\zeta = 1/\sqrt{K}$): ζ falls as K rises.
- Conflict: the error wants K up, the settling wants $\zeta\omega_n$ up, which the fixed s-coefficient forbids.
- Resolution: add rate feedback or a lead/PD compensator – it multiplies the effective s-coefficient without adding an integrator, meeting both.