

## 10.3 Time-domain response

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

**Total: 24 marks****KEY WORDS** overshoot • settling time • steady-state error

### Objective

Compute step-response specifications from the second-order prototype, map pole locations to transient behaviour, and read steady-state error constants off the loop.

### 1 Key knowledge and worked examples

#### ■ The prototype formulas

$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ ,  $t_p = \pi/(\omega_n\sqrt{1-\zeta^2})$ ,  $t_s \approx 4/(\zeta\omega_n)$  (2% band). Poles at  $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ : real part decays, imaginary part rings, distance from origin is  $\omega_n$ .

#### ■ Worked example: one number, three specs

$\omega_n = 6$ ,  $\zeta = 0.4$ :  $M_p = 25.4\%$ ,  $t_p = 0.571s$ ,  $t_s = 4/2.4 = 1.67s$  – and poles at  $-2.4 \pm j5.51$  (check:  $|pole| = \sqrt{2.4^2 + 5.51^2} = 6$  ✓).

#### ■ Error constants

Type 0:  $K_p = G(0)$ , step error  $1/(1 + K_p)$ . Type 1:  $K_v = \lim_{s \rightarrow 0} sG$ , ramp error  $1/K_v$ ; step error zero. Each integrator buys one order of tracking and costs phase.

### 2 Practice

**2.1** A system has  $\omega_n = 6 \text{ rad/s}$ ,  $\zeta = 0.4$ . Compute  $M_p$ ,  $t_p$  and  $t_s$  (2%). [3]

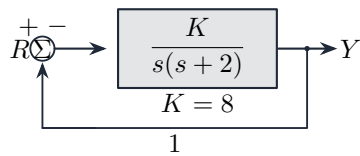
**2.2** Poles at  $-2 \pm j4$ : compute  $\omega_n$ ,  $\zeta$  and state whether the step response overshoots. [3]

**2.3** Compute the dc gain and the unit-step steady-state error for  $G = 5/(s + 2)$  in unity feedback. [3]

### 3 Exam-style questions

**3.1** A unity-feedback system has loop  $G = K/[s(s + 2)]$  with  $K = 8$ .

Write the closed-loop transfer function and its  $\omega_n$  and  $\zeta$ . Compute  $M_p$ ,  $t_p$  and  $t_s$ . Compute the unit-ramp steady-state error via  $K_v$ . Determine the  $K$  that makes  $\zeta = 0.5$  exactly, and verify. [8]



**3.2** A servo must track a ramp with error at most 0.1 and settle (2%) within 2 s. The plant is  $G = K/[s(s + 2)]$ .

Derive the ramp-error requirement on  $K_v$  and hence the minimum  $K$ . Derive the settling-time requirement on  $\zeta\omega_n$  and check it at that  $K$ . If the two requirements conflict, state which gain each wants and propose the standard resolution (a compensator, not a compromise gain). [7]

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## Solutions

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### 2.1

- $M_p = e^{-\pi \cdot 0.4 / 0.9165} = 25.4\%$ .
- $t_p = \pi / (6 \times 0.9165) = 0.571s$ .
- $t_s = 4 / (0.4 \times 6) = 1.67s$ .

### 2.2

- $\omega_n = |\text{pole}| = \sqrt{4 + 16} = 4.47 \text{ rad/s}$ .
- $\zeta = 2 / 4.47 = 0.447$  – underdamped, so yes it overshoots ( $M_p \approx 20.8\%$ ).

### 2.3

- $K_p = G(0) = 5/2 = 2.5$ .
- $e_{ss} = 1 / (1 + K_p) = 1/3.5 = 0.286$ .

### 3.1

- $T = 8 / (s^2 + 2s + 8)$ :  $\omega_n = 2.83 \text{ rad/s}$ ,  $\zeta = 2 / (2 \times 2.83) = 0.354$ .
- $M_p = e^{-\pi \cdot 0.354 / 0.935} = 30.5\%$ ;  $t_p = \pi / (2.83 \times 0.935) = 1.19s$ ;  $t_s = 4 / (0.354 \times 2.83) = 4.0s$ .
- $K_v = \lim_{s \rightarrow 0} sG(s) = K/2 = 4 \rightarrow \text{ramp error } 1/4 = 0.25$ .
- $\zeta = 0.5$  with the s coefficient fixed at 2:  $\omega_n = \sqrt{K}$  and  $\zeta = 2 / (2\sqrt{K})$ , so  $\zeta = 0.5$  needs  $\sqrt{K} = 2$ , giving  $K = 4$ . Check:  $s^2 + 2s + 4 \rightarrow \zeta = 2 / (2 \times 2) = 0.5$  exactly.

### 3.2

- Ramp error  $= 1/K_v = 2/K \leq 0.1 \rightarrow K \geq 20$ .
- $\zeta\omega_n = 2/2 = 1$  always (the s coefficient is 2)  $\rightarrow t_s = 4/1 = 4s$  at ANY gain of this plant.
- At  $K = 20$  the ramp spec holds but settling is  $4s > 2s$  – and raising  $K$  only worsens damping ratio ( $\zeta = 1/\sqrt{K}$ ):  $\zeta$  falls as  $K$  rises.
- Conflict: the error wants  $K$  up, the settling wants  $\zeta\omega_n$  up, which the fixed s-coefficient forbids.
- Resolution: add rate feedback or a lead/PD compensator – it multiplies the effective s-coefficient without adding an integrator, meeting both.