10.1 Modelling of dynamic systems

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS linearization • operating point

Objective

Derive first-order models from electrical, mechanical and thermal systems, linearize about an operating point, and read the physical meaning of the states.

1 Key knowledge and worked examples

■ **One skeleton, every domain**

Storage, flow, dissipation: $C \cdot dv/dt = (v_{in} - v)/R$ (RC), $J \cdot d\omega/dt = T - B\omega$ (rotor), $C \cdot dT/dt = P - hT$ (thermal). Each gives a time constant and a final value with no differential-equation machinery.

■ **Worked example: the DC motor**

Torque $K \cdot i$, back-EMF $K \cdot \omega$, J = 0.01, B = 0.1, R = 1, L negligible: $\omega/V = K/[(Js + B)R + K^2] = 50/(s + 35)$ – $\tau = 28.6$ ms, 1.43 rad/s per volt.

■ **Worked example: linearize the square root**

Tank outflow $q = k\sqrt{h}$ at $\bar{h} = 4$, $k = 2$: $\delta q \approx (k/(2\sqrt{\bar{h}})) \cdot \delta h = 0.5 \cdot \delta h$; with A = 1 m^2, $\delta h' = \delta q_{in} - 0.5\delta h$ – a first-order tank with $\tau = 2$ s. The small-signal gain **is** the slope at the operating point.

2 Practice

2.1 An RC network has R = 2 kΩ, C = 100 µF. Compute the time constant and the final capacitor voltage for a 9 V step. [3]

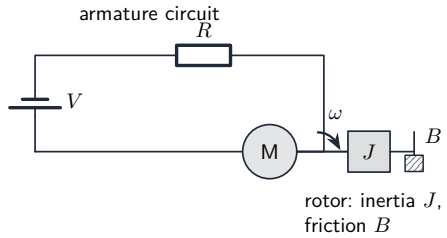
2.2 A thermal system: C = 200 J/K, h = 4 W/K, P = 100 W. Compute the time constant and the final temperature rise. [3]

2.3 A pendulum of length 0.4 m is linearised about straight-down. Compute its natural frequency in rad/s. [3]

3 Exam-style questions

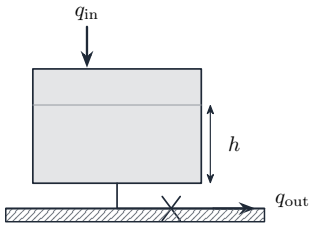
3.1 An armature-controlled DC motor has J = 0.02 kg · m^2, B = 0.2 N · m · s, K = 0.4 V · s/rad, R = 2 Ω (inductance negligible). The input is armature voltage V, the output is speed ω.

Write the two first-order equations (electrical and mechanical, with the back-EMF coupling). Eliminate the current to find $\omega(s)/V(s)$ as a single first-order transfer function. Compute the time constant and the steady state speed per volt. State the physical meaning and units of both state variables (i, ω). [8]



3.2 A mixing tank of area 1 m^2 drains through a valve with outflow $q_{\text{out}} = 2\sqrt{h}$ (m^3/h with h in m) about the operating point $\bar{h} = 4$ m; the inflow is q_{in}.

Linearise the outflow about $\bar{h}$ and write the small-signal model for δh. Compute the linearised resistance (slope of q vs h) and the tank's time constant. Compute the new time constant if the operating point moves to $\bar{h} = 9$ m, and explain physically why a deeper tank responds more slowly. [7]



Solutions

2.1

- $\tau = RC = 2000 \times 100\mu = 0.2s$.
- Final $v_C = 9V$ (capacitor is an open circuit at dc).

2.2

- $\tau = C/h = 200/4 = 50s$.
- $\Delta T_{final} = P/h = 100/4 = 25°C$ above ambient.

2.3

- $\omega_n = \sqrt{g/l} = \sqrt{9.81/0.4} = 4.95 rad/s$.

3.1

- Electrical (L = 0): $V = Ri + K\omega$; mechanical: $J\omega' + B\omega = Ki$.
- Eliminate: $i = (V - K\omega)/R \rightarrow (Js + B)\omega = K(V - K\omega)/R \rightarrow [(Js + B)R + K^2]\omega = KV$.
- $\omega/V = K/[(Js + B)R + K^2] = 0.4/(0.04s + 0.56) = 10/(s + 14)$.
- $\tau = 1/14 = 71.4$ ms.
- Final speed per volt $= 10/14 = 0.714$ rad/s/V.
- States: i the armature current (A, the energy-converting quantity) and ω the shaft speed (rad/s, the stored kinetic energy $J\omega^2/2$).

3.2

- $q_{out} \approx 2\sqrt{4} + (2/(2\sqrt{4})) \cdot \delta h = 4 + 0.5 \cdot \delta h$.
- $A \cdot \delta h' = \delta q_{in} - 0.5 \cdot \delta h \rightarrow$ with A = 1: $\delta h' + 0.5\delta h = \delta q_{in}$.
- Linearised resistance (level change per flow change): $1/0.5 = 2h/m^2$ – the slope is 0.5 $(m^3/h)/m$.
- τ = A/slope $= 1/0.5 = 2$ h.
- At $\bar{h} = 9$: slope $= 2/(2 \cdot 3) = 1/3 \rightarrow \tau = 3$ h.
- Physically: at greater depth the same δh changes the outflow less (the $\surd$ curve flattens), so the tank's self-regulation weakens and level excursions last longer.