

10.1 Modelling of dynamic systems

Name: _____ Class: _____ Date: _____

Total: 24 marks

KEY WORDS linearization • operating point

Objective

Derive first-order models from electrical, mechanical and thermal systems, linearize about an operating point, and read the physical meaning of the states.

1 Key knowledge and worked examples

■ One skeleton, every domain

Storage, flow, dissipation: $C \cdot dv/dt = (v_{in} - v)/R$ (RC), $J \cdot d\omega/dt = T - B\omega$ (rotor), $C \cdot dT/dt = P - hT$ (thermal). Each gives a time constant and a final value with no differential-equation machinery.

■ Worked example: the DC motor

Torque $K \cdot i$, back-EMF $K \cdot \omega$, $J = 0.01$, $B = 0.1$, $R = 1$, L negligible: $\omega/V = K/[(Js + B)R + K^2] = 50/(s + 35) - \tau = 28.6 \text{ ms}$, $1.43 \text{ rad/s per volt}$.

■ Worked example: linearize the square root

Tank outflow $q = k\sqrt{h}$ at $\bar{h} = 4$, $k = 2$: $\delta q \approx (k/(2\sqrt{\bar{h}})) \cdot \delta h = 0.5 \cdot \delta h$; with $A = 1 \text{ m}^2$, $\delta h' = \delta q_{in} - 0.5\delta h$ – a first-order tank with $\tau = 2 \text{ s}$. The small-signal gain is the slope at the operating point.

2 Practice

2.1 An RC network has $R = 2 \text{ k}\Omega$, $C = 100 \text{ }\mu\text{F}$. Compute the time constant and the final capacitor voltage for a 9 V step. [3]

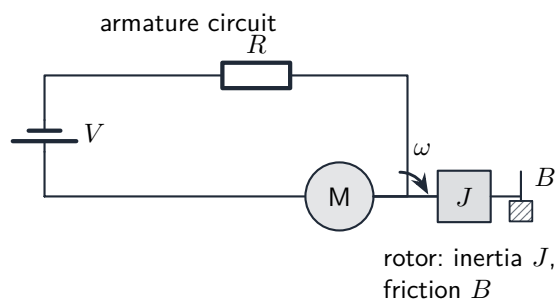
2.2 A thermal system: $C = 200 \text{ J/K}$, $h = 4 \text{ W/K}$, $P = 100 \text{ W}$. Compute the time constant and the final temperature rise. [3]

2.3 A pendulum of length 0.4 m is linearised about straight-down. Compute its natural frequency in rad/s . [3]

3 Exam-style questions

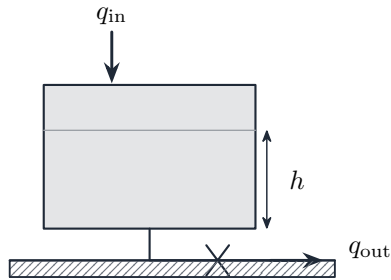
3.1 An armature-controlled DC motor has $J = 0.02 \text{ kg} \cdot \text{m}^2$, $B = 0.2 \text{ N} \cdot \text{m} \cdot \text{s}$, $K = 0.4 \text{ V} \cdot \text{s/rad}$, $R = 2 \Omega$ (inductance negligible). The input is armature voltage V , the output is speed ω .

Write the two first-order equations (electrical and mechanical, with the back-EMF coupling). Eliminate the current to find $\omega(s)/V(s)$ as a single first-order transfer function. Compute the time constant and the steady state speed per volt. State the physical meaning and units of both state variables (i , ω). [8]



3.2 A mixing tank of area 1 m^2 drains through a valve with outflow $q_{\text{out}} = 2\sqrt{h}$ (m^3/h with h in m) about the operating point $\bar{h} = 4 \text{ m}$; the inflow is q_{in} .

Linearise the outflow about $\bar{h}$ and write the small-signal model for δh . Compute the linearised resistance (slope of q vs h) and the tank's time constant. Compute the new time constant if the operating point moves to $\bar{h} = 9 \text{ m}$, and explain physically why a deeper tank responds more slowly. [7]



Solutions

2.1

- $\tau = RC = 2000 \times 100\mu = 0.2s$.
- Final $v_C = 9V$ (capacitor is an open circuit at dc).

2.2

- $\tau = C/h = 200/4 = 50s$.
- $\Delta T_{final} = P/h = 100/4 = 25^\circ C$ above ambient.

2.3

- $\omega_n = \sqrt{g/l} = \sqrt{9.81/0.4} = 4.95rad/s$.

3.1

- Electrical ($L = 0$): $V = Ri + K\omega$; mechanical: $J\omega' + B\omega = Ki$.
- Eliminate: $i = (V - K\omega)/R \rightarrow (Js + B)\omega = K(V - K\omega)/R \rightarrow [(Js + B)R + K^2]\omega = KV$.
- $\omega/V = K/[(Js + B)R + K^2] = 0.4/(0.04s + 0.56) = 10/(s + 14)$.
- $\tau = 1/14 = 71.4$ ms.
- Final speed per volt = $10/14 = 0.714$ rad/s/V.
- States: i the armature current (A, the energy-converting quantity) and ω the shaft speed (rad/s, the stored kinetic energy $J\omega^2/2$).

3.2

- $q_{out} \approx 2\sqrt{4} + (2/(2\sqrt{4})) \cdot \delta h = 4 + 0.5 \cdot \delta h$.
- $A \cdot \delta h' = \delta q_{in} - 0.5 \cdot \delta h \rightarrow$ with $A = 1$: $\delta h' + 0.5\delta h = \delta q_{in}$.
- Linearised resistance (level change per flow change): $1/0.5 = 2h/m^2$ – the slope is 0.5 (m^3/h)/m.
- $\tau = A/\text{slope} = 1/0.5 = 2$ h.
- At $\bar{h} = 9$: slope = $2/(2 \cdot 3) = 1/3 \rightarrow \tau = 3$ h.
- Physically: at greater depth the same δh changes the outflow less (the $\sqrt{}$ curve flattens), so the tank's self-regulation weakens and level excursions last longer.