Exercise walk-through

7.1
Vector field tools

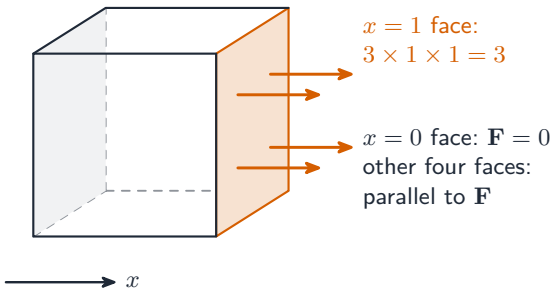
University Electrical Engineering

Vector tools: symmetry picks the surface

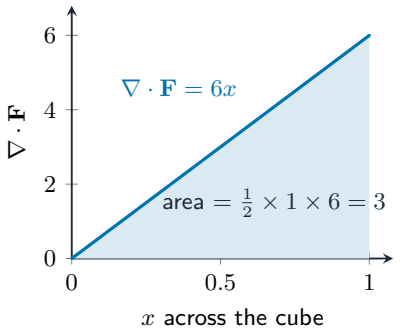
1. **Box** F = (3x^2,0,0): div 6x; cube flux 3 by volume integral AND by the two x-faces - identical.
2. **Curl** F = (-y,x,0): **curl** (0,0,2); unit-circle circulation 2pi = 2 x area - Stokes verified.
3. **Rule** No symmetry: integrate directly; the theorem checks the bookkeeping, never the physics.

Two routes, one number

(1) surface route: flux $= 3$



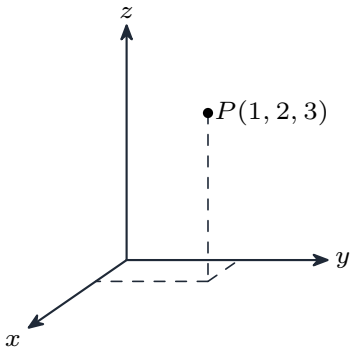
(2) volume route: integral $= 3$



Face flux 3 equals volume integral 6 times 0.5 $= 3$ - the **divergence** theorem as bookkeeping on a unit cube.

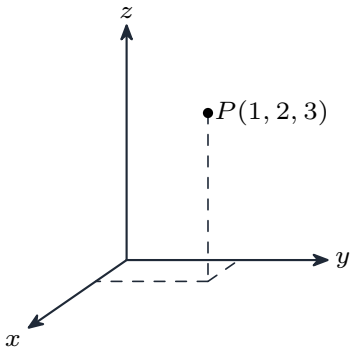
Exercise 2.1

2.1 For $\mathbf{F} = (xy,\ yz,\ zx)$, find $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ at the point $P(1, 2, 3)$. [3]



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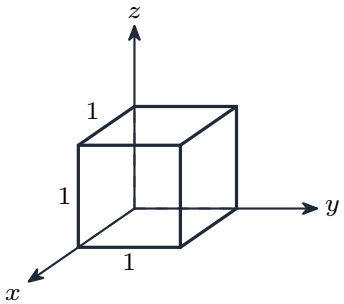


2.1

$\nabla \cdot \mathbf{F} = y + z + x$; at P: $2 + 3 + 1 = \mathbf{6}$; $\nabla \times \mathbf{F} = (0 - y,\ 0 - z,\ 0 - x) = (\mathbf{-2}, \mathbf{-3}, \mathbf{-1})$ at P

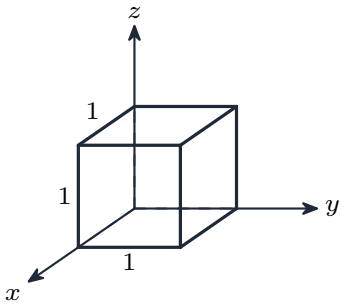
Exercise 2.2

2.2 For $\mathbf{F} = (x,\ y,\ 0)$, find the flux out of the unit cube shown in two ways: by the volume integral and face by face. [3]



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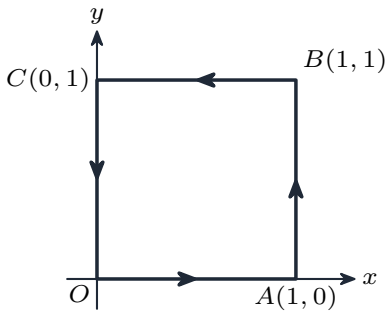
$\nabla \cdot \mathbf{F} = 1 + 1 + 0 = 2$, so the volume integral is $2 \times 1 = \mathbf{2}$

faces $x = 1$ and $y = 1$ each give $1 \times 1 = 1$ out; the faces $x = 0$, $y = 0$ have zero field normal to them and the z faces have none: total **2**, the same

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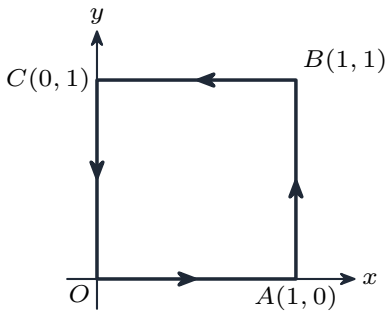
Exercise 2.3

2.3 For $\mathbf{F} = (y,\ 0,\ 0)$, find the circulation round the square $OABC$ shown, anticlockwise. Check your answer with Stokes' theorem. [3]



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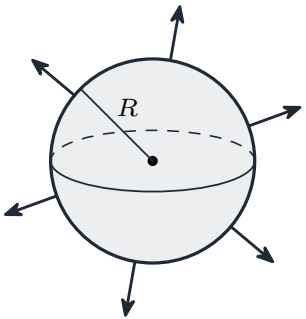
2.3

OA: $y = 0$, so 0; AB and CO: $d\mathbf{l}$ along y and $F_y = 0$, so 0; BC: $F_x = 1$ with $dx = -1$ in total, so -1; circulation $= \mathbf{-1}$

$(\nabla \times \mathbf{F})_z = \dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y} = -1$, times the area 1 gives -1

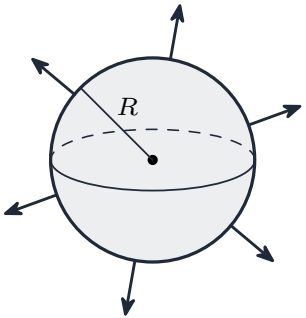
Exercise 2.4

2.4 The field $\mathbf{F} = r\,\hat{\mathbf{r}}$ points straight out from the origin, with size r. Find its flux out of the sphere of radius $R = 2$ shown. Check with the divergence theorem. [3]



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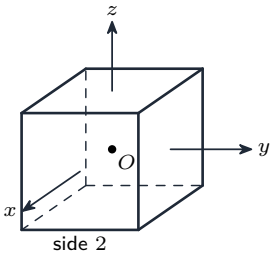
on the sphere $F = 2$ everywhere and is normal to it: flux $= 2 \times 4\pi(2)^2 = \mathbf{32}\pi$

$\nabla \cdot \mathbf{F} = \dfrac{1}{r^2}\dfrac{d(r^3)}{dr} = 3$; $3 \times \frac{4}{3}\pi(2)^3 = 32\pi$, the same

Exercise 3.1

3.1 The field is $\mathbf{F} = (2x,\ -2y,\ 2z)$. The cube shown is $-1 \le x, y, z \le 1$.

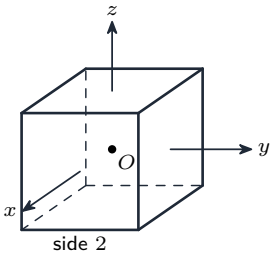
Find $\nabla \cdot \mathbf{F}$ and the flux out of the cube by the volume integral. Check it face by face. Find $\nabla \times \mathbf{F}$, and say in one sentence what the field looks like. [5]



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3.1

$\nabla \cdot \mathbf{F} = 2 - 2 + 2 = 2$; volume integral $2 \times 8 = \mathbf{16}$

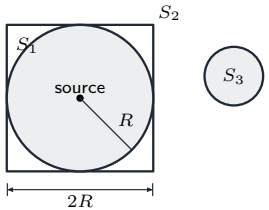
face $x = +1$: $F_x = 2$ over area 4, so $+8$; face $x = -1$: $F_x = -2$ with outward normal $-\hat{\mathbf{x}}$, so $+8$; the y faces give -8 each (inward) and the z faces $+8$ each: $8+8-8-8+8+8 = \mathbf{16}$

$\nabla \times \mathbf{F} = \mathbf{0}$; the field flows out along $\pm x$ and $\pm z$ and in along $\pm y$, with no rotation anywhere

Exercise 3.2

3.2 A point source at the origin has the field $\mathbf{F} = \dfrac{k}{r^2}\hat{\mathbf{r}}$. The figure shows three closed surfaces: a sphere S_1 of radius R round the origin, a cube S_2 of side $2R$ round the origin, and a small sphere S_3 that does not enclose the origin.

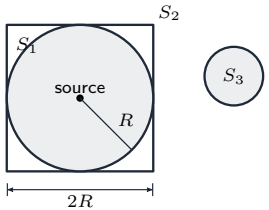
Find the flux out of S_1 and explain why it does not depend on R. Find $\nabla \cdot \mathbf{F}$ for $r \neq 0$. Use it to find the flux out of S_2 and out of S_3. [5]



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3.2

on S_1: $\dfrac{k}{R^2} \times 4\pi R^2 = \mathbf{4\pi k}$; the field falls as $\dfrac{1}{r^2}$ while the area grows as r^2, so they cancel

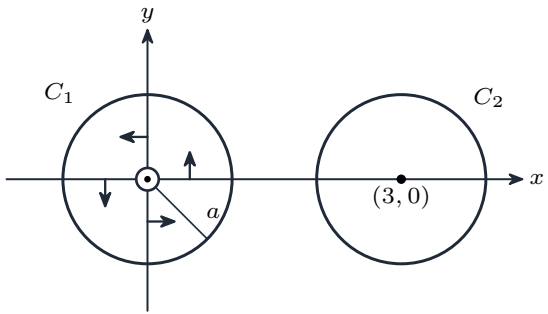
$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{k}{r^2}\right) = \mathbf{0} \text{ for } r \neq 0$$

the region between S_1 and S_2 has no sources, so S_2 also carries $\mathbf{4\pi k}$; S_3 encloses no source, so its flux is $\mathbf{0}$

Exercise 3.3 (a)

3.3 Outside a long straight wire on the z-axis, the magnetic field has the shape $\mathbf{F} = \dfrac{(-y,\ x,\ 0)}{x^2 + y^2}$. In the figure, the loop C_1 is a circle of radius a round the wire, and C_2 is a circle of radius 1 centred at $(3, 0)$.

(a) Show that $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere except on the wire. [2]

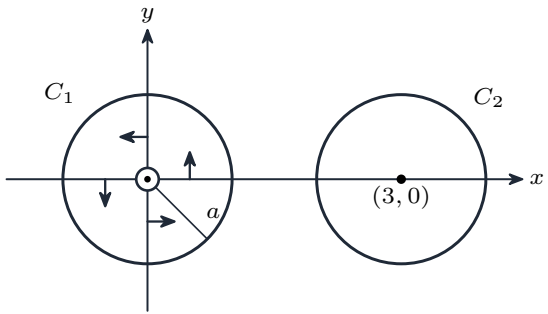


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(a) $\dfrac{\partial F_y}{\partial x} = \dfrac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}$ and $\dfrac{\partial F_x}{\partial y} = -\dfrac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}$; their difference is $\dfrac{2(x^2+y^2)-2z}{(x^2+y^2)}$ 0; the x and y parts are zero because nothing depends on z

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(b) on C_1, $|\mathbf{F}| = \dfrac{1}{a}$ along the path, so $\oint = \dfrac{1}{a} \times 2\pi a = \mathbf{2}\pi$, the same for every a

Exercise 3.3 (c)

3.3 Outside a long straight wire on the z-axis, the magnetic field has the shape $\mathbf{F} = \dfrac{(-y,\ x,\ 0)}{x^2 + y^2}$. In the figure, the loop C_1 is a circle of radius a round the wire, and C_2 is a circle of radius 1 centred at $(3, 0)$.

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(c) C_2 does not enclose the wire, so the curl is zero over the whole disc inside it; by Stokes' theorem the circulation is **0**

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Exercise 3.3 (d)

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(d) A **Rogowski coil** measures the circulation of $\mathbf{B}$ round any closed loop it is bent into. Explain why its reading does not depend on where the wire sits inside the loop, and what it reads for a wire outside the loop. [2]

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(d) every loop round the wire gives the same circulation, $\mu_0 I$, whatever its shape or the wire's position, because the curl is zero away from the wire; a wire outside the loop gives zero, so the coil measures only the current it encloses